

# UNIVERSAL INDEX THEOREM ON $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$

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**ABSTRACT.** By conformal welding, there is a pair of univalent functions  $(f, g)$  associated to every point of the complex Kähler manifold  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ . For every integer  $n \geq 1$ , we generalize the definition of Faber polynomials to define some canonical bases of holomorphic  $1-n$  and  $n$  differentials associated to the pair  $(f, g)$ . Using these bases, we generalize the definition of Grunsky matrices to define matrices whose columns are the coefficients of the differentials with respect to standard bases of differentials on the unit disc and the exterior unit disc. We derive some identities among these matrices which are reminiscent of the Grunsky equality. By using these identities, we showed that we can define the Fredholm determinants of the period matrices of holomorphic  $n$  differentials  $N_n$ , which are the Gram matrices of the canonical bases of holomorphic  $n$ -differentials with respect to the inner product given by the hyperbolic metric. Finally we proved that  $\det N_n = (\det N_1)^{6n^2-6n+1}$  and  $\partial\bar{\partial} \log \det N_n$  is  $-(6n^2 - 6n + 1)/(6\pi i)$  of the Weil-Petersson symplectic form.

## 1. INTRODUCTION

This paper can be considered as a sequel to our paper [TT06], in which we study the properties of the Weil-Petersson metric on the universal Teichmüller space. However, a fundamental difference is that in this paper we no longer work with the entire universal Teichmüller space, but instead we work with its subspace  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , which corresponds to smooth  $(C^\infty)$  mappings. As a result, we make no use of any quasi-conformal mapping theories, and we hope that this paper may be more accessible to people working in theoretical physics.

Let  $\mathbb{D}$  and  $\mathbb{D}^*$  denotes the unit disc and its exterior respectively. In [Kir87], it was shown that every element  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$  is associated with a unique pair of univalent functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  and  $g : \mathbb{D}^* \rightarrow \hat{\mathbb{C}}$  such that  $\gamma = g^{-1} \circ f$ ,  $f(0) = 0, g(\infty) = \infty$  and  $f'(0) = 1$ . The (modified)<sup>1</sup>Faber polynomials  $u[0]_k(w)$  and  $v[0]_k(w)$ ,  $k \geq 1$  of  $(f, g)$  (see e.g. [Pom75, Dur83, Teo03]) is defined by

$$(1.1) \quad u[0]_k(w) = \frac{1}{\sqrt{\pi k}} (g^{-1}(w))_{\geq 1}, \quad v[0]_k(w) = \frac{1}{\sqrt{\pi k}} (f^{-1}(w))_{\leq -1}.$$

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<sup>1</sup>In order to conform with our later definitions, we modify slightly the definition, which differs from the usual one by some constants.

They can be encoded in

$$\begin{aligned}\frac{g'(z)w}{(g(z)-w)g(z)} &= \sum_{k=1}^{\infty} u[0]_k(w) \sqrt{\pi k} z^{-k-1}, \\ \frac{f'(z)}{f(z)-w} &= - \sum_{k=1}^{\infty} v[0]_k(w) \sqrt{\pi k} z^{k-1}.\end{aligned}$$

Define the Grunsky coefficients  $b_{mn}$  of  $(f, g)$  [Pom75, Dur83, Teo03] by

$$\begin{aligned}\log \frac{g(z) - g(\zeta)}{z - \zeta} &= b_{0,0} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} z^{-m} w^{-n}, \\ \log \frac{g(z) - f(\zeta)}{z} &= b_{0,0} - \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{m,-n} z^{-m} \zeta^n, \\ \log \frac{f(z) - f(\zeta)}{z - \zeta} &= b_{0,0} - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{-m,-n} z^m \zeta^n,\end{aligned}$$

and for  $m \geq 0, n \geq 1, b_{-m,n} = b_{m,-n}$ . Then

$$\begin{aligned}u[0]_k(f(z)) &= \sum_{l=1}^{\infty} \sqrt{k l} b_{-l,k} \frac{1}{\sqrt{\pi l}} z^l, \\ u[0]_k(g(z)) &= \frac{1}{\sqrt{\pi k}} z^k - \sqrt{\frac{k}{\pi}} b_{0,k} + \sum_{l=1}^{\infty} \sqrt{k l} b_{l,k} \frac{1}{\sqrt{\pi l}} z^{-l}, \\ v[0]_k(f(z)) &= \frac{1}{\sqrt{\pi k}} z^{-k} + \sqrt{\frac{k}{\pi}} b_{0,-k} + \sum_{l=1}^{\infty} \sqrt{k l} b_{-l,-k} \frac{1}{\sqrt{\pi l}} z^l, \\ v[0]_k(g(z)) &= \sum_{l=1}^{\infty} \sqrt{k l} b_{l,-k} \frac{1}{\sqrt{\pi l}} z^{-l}.\end{aligned}$$

The Grunsky matrices  $A[0], B[0], C[0], D[0]$  are semi-infinite matrices defined by

$$\begin{aligned}A[0] &= (\sqrt{k l} b_{-l,k})_{l,k \geq 1}, & B[0] &= (\sqrt{k l} b_{l,k})_{l,k \geq 1}, \\ C[0] &= (\sqrt{k l} b_{-l,-k})_{l,k \geq 1}, & D[0] &= (\sqrt{k l} b_{l,-k})_{l,k \geq 1}.\end{aligned}$$

Grunsky equality [Hum72] (or see [TT06]) says that for any complex numbers  $\lambda_k$ ,  $k = \pm 1, \dots, \pm m$ ,

$$\sum'_{k=-\infty}^{\infty} \left| \sum'_{l=-m}^m \sqrt{|k l|} b_{kl} \lambda_l \right|^2 = \sum'_{k=-m}^m |\lambda_k|^2.$$

In other words,

$$(1.2) \quad \begin{pmatrix} B[0] & D[0] \\ A[0] & C[0] \end{pmatrix} \begin{pmatrix} B[0]^* & A[0]^* \\ D[0]^* & C[0]^* \end{pmatrix} = \text{Id}.$$

In particular,

$$A[0]A[0]^* + C[0]C[0]^* = \text{Id}, \quad D[0]D[0]^* + B[0]B[0]^* = \text{Id},$$

which implies that both  $A[0]$  and  $D[0]$  define bounded operators on  $\ell^2$  with norm less than or equal to one. In [TT06], we defined a basis for  $A_{1,2}(\Omega)$  and  $A_{1,2}(\Omega^*)$

– the Hilbert spaces of square-integrable holomorphic one-forms on  $\Omega = f(\mathbb{D})$  and  $\Omega^* = g(\mathbb{D}^*)$  – by

$$(1.3) \quad \left\{ \begin{aligned} U[1]_k(w) : U[1]_k &= u[0]'_k(w) = \sqrt{\frac{k}{\pi}} \left( g^{-1}(w)^{k-1} (g^{-1})'(w) \right)_{\geq 0}, \quad k \geq 1 \end{aligned} \right\},$$

$$\left\{ \begin{aligned} V[1]_k(w) : V[1]_k(w) &= -v[0]'_k(w) = \sqrt{\frac{k}{\pi}} \left( f^{-1}(w)^{-k-1} (f^{-1})'(w) \right)_{\leq -2}, \quad k \geq 1 \end{aligned} \right\}.$$

Then

$$\begin{aligned} U[1]_k(f(z))f'(z) &= \sum_{l=1}^{\infty} \sqrt{k} l b_{-l,k} \sqrt{\frac{l}{\pi}} z^{l-1}, \\ U[1]_k(g(z))g'(z) &= \sqrt{\frac{k}{\pi}} z^{k-1} - \sum_{l=1}^{\infty} \sqrt{k} l b_{l,k} \sqrt{\frac{l}{\pi}} z^{-l-1}, \\ V[1]_k(f(z))f'(z) &= \sqrt{\frac{k}{\pi}} z^{-k-1} - \sum_{l=1}^{\infty} \sqrt{k} l b_{-l,-k} \sqrt{\frac{l}{\pi}} z^{l-1}, \\ V[1]_k(g(z))g'(z) &= \sum_{l=1}^{\infty} \sqrt{k} l b_{l,-k} \sqrt{\frac{l}{\pi}} z^{l-1}. \end{aligned}$$

The matrices  $A[1], B[1], C[1], D[1]$  are defined by

$$\begin{aligned} A[1] &= (\sqrt{k} l b_{-l,k})_{l,k \geq 1}, & B[1] &= (-\sqrt{k} l b_{l,k})_{l,k \geq 1}, \\ C[1] &= (-\sqrt{k} l b_{-l,-k})_{l,k \geq 1}, & D[1] &= (\sqrt{k} l b_{l,-k})_{l,k \geq 1}, \end{aligned}$$

so that their columns correspond to the coefficients of  $U[1]_k(f(z))f'(z)$ ,  $U[1]_k(g(z))g'(z)$ ,  $V[1]_k(f(z))f'(z)$  and  $V[1]_k(g(z))g'(z)$  with respect to the standard bases

$$\left\{ \sqrt{\frac{k}{\pi}} z^{k-1} : k \geq 1 \right\} \quad \text{and} \quad \left\{ \sqrt{\frac{k}{\pi}} z^{-k-1} : k \geq 1 \right\}$$

of  $A_{1,2}(\mathbb{D})$  and  $A_{1,2}(\mathbb{D}^*)$ . Obviously,  $A[1] = A[0]$ ,  $B[1] = -B[0]$ ,  $C[1] = -C[0]$  and  $D[1] = D[0]$ . The Gram matrices of the bases  $\{U[1]_k : k \geq 1\}$  and  $\{V[1]_k : k \geq 1\}$  with respect to the inner product is given by

$$(N_1(\Omega)_{lk})_{l,k \geq 1} = \left( \iint_{\Omega} U[1]_l(w) \overline{U[1]_k(w)} d^2 w \right)_{l,k \geq 1} = A[1]^T \overline{A[1]} = D[1] D[1]^*$$

and

$$(N_1(\Omega^*)_{lk})_{l,k \geq 1} = \left( \iint_{\Omega^*} V[1]_l(w) \overline{V[1]_k(w)} d^2 w \right)_{l,k \geq 1} = D[1]^T \overline{D[1]} = A[1] A[1]^*$$

respectively. We call  $N_1(\Omega)$  and  $N_1(\Omega^*)$  the period matrices of holomorphic one-forms.

Define the kernels

$$\begin{aligned}\mathcal{A}[1](z, w) &= \frac{1}{\pi} \frac{f'(z)g'(w)}{(f(z) - g(w))^2} = \frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k l b_{-l, k} z^{l-1} w^{-k-1}, \\ \mathcal{B}[1](z, w) &= \frac{1}{\pi} \left( \frac{g'(z)g'(w)}{(g(z) - g(w))^2} - \frac{1}{(z - w)^2} \right) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k l b_{l, k} z^{-k-1} w^{-l-1}, \\ \mathcal{C}[1](z, w) &= \frac{1}{\pi} \left( \frac{f'(z)f'(w)}{(f(z) - f(w))^2} - \frac{1}{(z - w)^2} \right) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k l b_{-l, -k} z^{k-1} w^{l-1}, \\ \mathcal{D}[1](z, w) &= \frac{1}{\pi} \frac{g'(z)f'(w)}{(g(z) - f(w))^2} = \frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k l b_{l, -k} z^{-l-1} w^{k-1}.\end{aligned}$$

They define operators  $\mathcal{A}[1] : \overline{A_{1,2}(\mathbb{D}^*)} \rightarrow A_{1,2}(\mathbb{D})$ ,  $\mathcal{B}[1] : \overline{A_{1,2}(\mathbb{D}^*)} \rightarrow A_{1,2}(\mathbb{D}^*)$ ,  $\mathcal{C}[1] : \overline{A_{1,2}(\mathbb{D})} \rightarrow A_{1,2}(\mathbb{D})$  and  $\mathcal{D}[1] : \overline{A_{1,2}(\mathbb{D})} \rightarrow A_{1,2}(\mathbb{D}^*)$  by

$$(\mathcal{A}[1]\phi)(z) = \iint_{\mathbb{D}} \mathcal{A}[1](z, w) \overline{\phi(w)} d^2 w$$

and similarly for  $\mathcal{B}[1]$ ,  $\mathcal{C}[1]$  and  $\mathcal{D}[1]$ . With respect to the standard bases of  $A_{1,2}(\mathbb{D})$  and  $A_{1,2}(\mathbb{D}^*)$ , their matrices are given by  $A[1]$ ,  $B[1]$ ,  $C[1]$  and  $D[1]$  respectively. We showed in [TT06] that for  $\gamma \in \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ ,  $\mathcal{B}[1]$  and  $\mathcal{C}[1]$  are Hilbert-Schmidt operators. Therefore by the Grunsky equality (1.2), the Fredholm determinants of the period matrices  $A[1]A[1]^*$  and  $D[1]D[1]^*$  are well defined. We defined  $\mathfrak{F}_1 : \text{Möb}(S^1) \setminus \text{Diff}_+(S^1) \rightarrow \mathbb{R}$  by

$$\mathfrak{F}_1(\gamma) = \log \det(A[1]A[1]^*) = \log \det(D[1]D[1]^*).$$

Since  $A[1]A[1]^*$  is the matrix of the operator  $\mathcal{A}[1]\mathcal{A}[1]^*$ ,  $\mathfrak{F}_1$  can also be interpreted as

$$\mathfrak{F}_1 = \log \prod_{l=1}^{\infty} \lambda_l,$$

where  $1 \geq \lambda_1 \geq \lambda_2 \geq \dots$  are the eigenvalues of the operator  $\mathcal{A}[1]\mathcal{A}[1]^*$ . In this aspect, this function has been considered in [Sch57].

In [Nag92] (see also [NS95]), Nag considered a period mapping on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  in the following way. For any  $\gamma \in \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  and  $k \neq 0$ , let

$$\frac{1}{\sqrt{\pi|k|}} \left( \gamma(z)^k - \frac{1}{2\pi i} \oint_{S^1} \gamma(\zeta)^k \frac{d\zeta}{\zeta} \right) = \sum_{l \in \mathbb{Z}} \Pi[0]_{lk} \frac{1}{\sqrt{\pi|l|}} z^l \quad \text{for } z = e^{i\theta} \in S^1,$$

and define

$$\Pi_1[0] = (\Pi[0]_{lk})_{l, k \geq 1}, \quad \Pi_2[0] = (\Pi[0]_{-l, k})_{l, k \geq 1}.$$

They satisfy the following identity:

$$(1.4) \quad \begin{pmatrix} \Pi_1[0] & \overline{\Pi_2[0]} \\ \Pi_2[0] & \overline{\Pi_1[0]} \end{pmatrix} \begin{pmatrix} \Pi_1[0]^* & -\Pi_2[0]^* \\ -\overline{\Pi_2[0]}^* & \overline{\Pi_1[0]}^* \end{pmatrix} = \text{Id}.$$

Nag defined the period mapping by

$$\gamma \mapsto \overline{\Pi_2[0]} \overline{\Pi_1[0]}^{-1}.$$

We proved that ([TT06])

$$\begin{aligned} A[0] &= (\Pi_1[0]^*)^{-1}, & B[0] &= -\Pi_2[0]^T (\Pi_1[0]^*)^{-1}, \\ C[0] &= \overline{\Pi_2[0]} \overline{\Pi_1[0]}^{-1}, & D[0] &= \overline{\Pi_1[0]}^{-1}, \end{aligned}$$

and therefore the period mapping can be equivalently defined by

$$\gamma \mapsto C[\gamma; 0],$$

which is the definition given by Kirillov and Yuriev [KY88], and (1.4) is equivalent to the Grunsky equality (1.2).

Given a smooth curve  $\gamma_t \in \text{Möb}(S^1) \backslash \text{Diff}_+(S^1)$  where  $\gamma_0 = \gamma$ , it defines a tangent vector at  $\gamma$  in the following way. Let  $u_t = \gamma_t \circ \gamma^{-1}$  and

$$\left. \frac{du_t}{dt} \right|_{t=0} (z) = \sum_{k \in \mathbb{Z}} c_k z^{k+1}, \quad c_{-k} = -\bar{c}_k, \quad z \in S^1.$$

It corresponds to the holomorphic and anti-holomorphic tangent vectors

$$v(z) = \sum_{k=2}^{\infty} c_k z^{k+1} \quad \text{and} \quad \bar{v}(z) = \overline{v(z)} = \sum_{k=2}^{\infty} \bar{c}_k \bar{z}^{k+1}.$$

In [TT06], we showed that the partial derivative of the function  $\mathfrak{F}_1$  is given by

$$\partial \mathfrak{F}_1(v) = \frac{1}{12\pi i} \oint_{S^1} \mathcal{S}(g)(z) v(z) dz,$$

where  $\mathcal{S}(g)(z)$  is the Schwarzian derivative of  $g$ . On the other hand, we proved that the function  $S : \text{Möb}(S^1) \backslash \text{Diff}_+(S^1) \rightarrow \mathbb{R}$  defined by

$$S(\gamma) = \iint_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 d^2 z + \iint_{\mathbb{D}^*} \left| \frac{g''(z)}{g'(z)} \right|^2 d^2 z - 4\pi \log |g'(\infty)|$$

satisfies

$$\partial S(v) = i \oint_{S^1} \mathcal{S}(g)(z) v(z) dz$$

and

$$\partial \bar{\partial} S(v, \bar{v}) = \|v\|_{WP}^2 = 2\pi \sum_{k=2}^{\infty} (k^3 - k) |c_k|^2.$$

This implies that  $S$  is a Weil-Petersson potential of  $\text{Möb}(S^1) \backslash \text{Diff}_+(S^1)$  and

$$(1.5) \quad \det N_1(\Omega) = \det A[1] A[1]^* = \exp \left( -\frac{1}{12\pi} S \right).$$

In [MT06b], inspired by the work of [MT06a], we showed that for a pair of Riemann surfaces  $X$  and  $Y$  of genus  $g$  which are uniformized simultaneously by a quasi-Fuchsian group  $\Gamma$ ,

$$(1.6) \quad \frac{\det \Delta_n}{\det N_n} = \exp \left( \frac{6n^2 - 6n + 1}{12\pi} S_{QF} \right) |F(n)|^2, \quad n \geq 2,$$

where  $\Delta_n$  is the  $n$ -Laplacian of the pair  $(X, Y)$ ,  $N_n$  is the Gram matrix of a basis of holomorphic  $n$ -differentials with respect to the inner product induced by hyperbolic metrics,  $S_{QF}$  is a Weil-Petersson potential of the quasi-Fuchsian deformation space

and  $F(n)$  is a function defined by the group elements of  $\Gamma$ . This formula is the anti-derivative of the local index theorem (see e.g. [TZ91]) which states that

$$(1.7) \quad \partial \bar{\partial} \log \det \Delta_n - \partial \bar{\partial} \log \det N_n = \frac{6n^2 - 6n + 1}{6\pi i} \omega_{WP},$$

where  $\omega_{WP}$  is the symplectic two-form corresponding to the Weil-Petersson metric. In physics notation, the term

$$\frac{6n^2 - 6n + 1}{6\pi i} \omega_{WP}$$

here is the anomaly term. Since pairs of Riemann surfaces  $(X, Y)$  simultaneously uniformized by quasi-Fuchsian groups and the homogeneous space  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  both sit inside the universal Teichmüller space, it is natural to look for the generalization of the index theorem (1.7) and the factorization formula (1.6) to  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , which is the main question addressed in this paper. In fact, (1.5) is the generalization we are looking for when  $n = 1$ .

To prove the formula (1.6), one of the necessary ingredients is to consider a basis of holomorphic  $n$ -differentials of  $X$  and  $Y$  defined by the Bers integral operator [Ber66]  $K[n] : \overline{A_{n,2}(Y)} \rightarrow A_{n,2}(X)$  :

$$(K[n]\phi)(z) = \iint_Y K[n](z, w) \overline{\phi(w)} \rho(w)^{1-n} d^2 w,$$

where with the coordinates provided by the covering maps  $J_1 : \Omega \rightarrow X$ ,  $J_2 : \Omega^* \rightarrow Y$ , the kernel  $K[n](z, w)$  is given by

$$K[n](z, w) = \frac{2^{2n-2}(2n-1)}{\pi} \sum_{\sigma \in \Gamma} \frac{\sigma'(w)^n}{(z - \sigma(w))^{2n}}, \quad z \in \Omega, w \in \Omega^*.$$

The term  $\det N_n$  in (1.6) turned up to be equal to  $\det K[n]K[n]^*$ . On the other hand, the Grunsky matrix  $A[1]$  can be considered as the generalization of  $K[n]$  to  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  and  $n = 1$ . This gives us some hints of how to generalize (1.6) to  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  for arbitrary  $n \geq 2$ .

In this paper, we start by working with the manifold  $S^1 \setminus \text{Diff}_+(S^1)$ , a fiber space of  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  with fiber isomorphic to  $\mathbb{D}^*$ . For any point  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$  and any integer  $n \geq 1$ , we define bases of holomorphic  $1-n$  differentials and  $n$  differentials for  $\Omega = f(\mathbb{D})$  and  $\Omega^* = g(\mathbb{D}^*)$ , which mimic the definitions of  $\{u[0]_k\}, \{v[0]_k\}$  (1.1) and  $\{U[1]_k, V[1]_k\}$  (1.3). In fact, there are two ways to generalize  $\{U[1]_k\}, \{V[1]_k\}$ , one of the them we denote by  $\{U[n]_k\}, \{V[n]_k\}$ , and the other by  $\{p[n]_k\}, \{q[n]_k\}$ . The bases  $\{p[n]_k\}, \{q[n]_k\}$  are related to the kernel of Bers integral operator  $K[n] : \overline{A_{n,2}(\Omega^*)} \rightarrow A_{n,2}(\Omega)$  by

$$\begin{aligned} & K[n](f(z), g(w)) f'(z)^n g'(w)^n \\ &= \frac{2^{2n-2}(2n-1)}{\pi} \frac{f'(z)^n g'(w)^n}{(f(z) - g(w))^{2n}} \\ &= \sum_{k=n}^{\infty} p[n]_k(f(z)) f'(z)^n \sqrt{\frac{2^{2n-2}}{\pi(2n-2)!} \frac{(k+n-1)!}{(k-n)!}} w^{-k-n} \\ &= \sum_{k=n}^{\infty} q[n]_k(g(w)) g'(w)^n \sqrt{\frac{2^{2n-2}}{\pi(2n-2)!} \frac{(k+n-1)!}{(k-n)!}} z^{k-n}. \end{aligned}$$

We denote the transition matrices from the basis  $\{U[n]_k\}$  to the basis  $\{p[n]_k\}$  and from the basis  $\{V[n]_k\}$  to the basis  $\{q[n]_k\}$  by  $\mathfrak{P}[n]$  and  $\mathfrak{M}[n]$  respectively. They are upper triangular matrices with diagonal elements identically equal to 1. We show that the matrices  $\mathfrak{P}[n] - \text{Id}[n]$  and  $\mathfrak{M}[n] - \text{Id}[n]$  are trace class. We also define the matrices  $A[1-n]$ ,  $B[1-n]$ ,  $C[1-n]$ ,  $D[1-n]$ ,  $A[n]$ ,  $B[n]$ ,  $C[n]$ ,  $D[n]$ ,  $\mathfrak{A}[n]$ ,  $\mathfrak{B}[n]$ ,  $\mathfrak{C}[n]$ ,  $\mathfrak{D}[n]$  so that their columns give the coefficients of the expansions of  $u[1-n]_k(f(z))f'(z)^{1-n}$ ,  $u[1-n]_k(g(z))g'(z)^{1-n}$ ,  $v[1-n]_k(f(z))f'(z)^{1-n}$ ,  $v[1-n]_k(g(z))g'(z)^{1-n}$ ,  $U[n]_k(f(z))f'(z)^n$ ,  $U[n]_k(g(z))g'(z)^n$ ,  $V[n]_k(f(z))f'(z)^n$ ,  $V[n]_k(g(z))g'(z)^n$ ,  $p[n]_k(f(z))f'(z)^n$ ,  $p[n]_k(g(z))g'(z)^n$ ,  $q[n]_k(f(z))f'(z)^n$ ,  $q[n]_k(g(z))g'(z)^n$  with respect to standard bases of corresponding differentials on  $\mathbb{D}$  and  $\mathbb{D}^*$ . Since  $\mathfrak{A}[n]$  is the matrix of  $K[n]$  with respect to the standard bases of  $n$ -differentials on  $\mathbb{D}$  and  $\mathbb{D}^*$ , by proving that the Bers integral operator  $K[n]$  is a bounded operator, we can conclude that these matrices all define bounded operators on some Hilbert spaces.

With the choice of the bases  $\{U[n]_k\}, \{V[n]_k\}$  or  $\{p[n]_k\}, \{q[n]_k\}$  of  $A_{n,2}(\mathbb{D})$  and  $A_{n,2}(\mathbb{D}^*)$ , we define the period matrices  $N_n(\Omega)$ ,  $N_n(\Omega^*)$  of holomorphic  $n$  differentials of  $\Omega$  and  $\Omega^*$  to be the Gram matrices of these bases with respect to the inner products on  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$  respectively. Depending on the bases we choose, they are equal to  $A[n]^T \overline{A[n]}$  and  $D[n]^T \overline{D[n]}$  in the former case, and equal to  $\mathfrak{A}[n]^T \overline{\mathfrak{A}[n]}$  and  $\mathfrak{D}[n]^T \overline{\mathfrak{D}[n]}$  in the later. One of our main challenge is to show that the function  $\mathfrak{F}_n : \text{Möb}(S^1) \setminus \text{Diff}_+(S^1) \rightarrow \mathbb{R}$ , where

$$\mathfrak{F}[n] = \log \det N_n(\Omega) = \log \det A[n]A[n]^*$$

is well-defined. Namely, we need to show that  $A[n]A[n]^* - \text{Id}[n]$  is of trace class. Since  $\mathfrak{A}[n] = A[n]\mathfrak{P}[n]$ , and  $\mathfrak{P}[n] - \text{Id}[n]$  is a strictly upper triangular trace class operator,  $\mathfrak{F}_n$  can also be interpreted as  $\log \det K[n]K[n]^*$ .

To show that  $A[n]A[n]^* - \text{Id}[n]$  is of trace class, we derive Grunsky-like equalities which generalize the Grunsky equality (1.2) for Grunsky matrices. We first define the  $\mathbb{Z} \times \mathbb{Z}$  matrix  $\Pi[\gamma; n]$  for any integer  $n$  so that its columns are, up to normalization constants, given by the coefficients of the expansion of  $\gamma(z)^{k-n}\gamma'(z)^n$ ,  $k \in \mathbb{Z}$  with respect to  $z^{l-n}$ . Let

$$\Pi_1[\gamma; n] = (\Pi[\gamma; n])_{l,k \geq n}, \quad \Pi_2[\gamma; n] = (\Pi[\gamma; n]_{-l,k})_{l \geq 1-n, k \geq n}.$$

We show that for  $n \geq 1$ , there exists matrices  $\mathfrak{S}_j[n]$ ,  $j = 1, 2, 3, 4$  such that

$$(1.8) \quad \begin{pmatrix} \Pi_1[\gamma; n] & \overline{\Pi_2[\gamma; n]} \\ \Pi_2[\gamma; n] & \overline{\Pi_1[\gamma; n]} \end{pmatrix} \begin{pmatrix} \Pi_1[\gamma; n]^* & -\Pi_2[\gamma; n]^* \\ -\Pi_2[\gamma; n]^* & \Pi_1[\gamma; n]^* \end{pmatrix} = \begin{pmatrix} \text{Id}[n] + \mathfrak{S}_1[n] & \mathfrak{S}_3[n] \\ \mathfrak{S}_2[n] & \text{Id}[n] + \mathfrak{S}_4[n] \end{pmatrix}$$

and  $\mathfrak{S}_1[n] = \overline{\mathfrak{S}_4[n]}$  is trace class. By showing that

$$\begin{aligned} A[\gamma; n] &= \Pi_1[\gamma^{-1}; n]^{-1}, & B[\gamma; n] &= \Pi_2[\gamma^{-1}; n]\Pi_1[\gamma^{-1}; n]^{-1}, \\ C[\gamma; n] &= \overline{\Pi_2[\gamma; n]}\Pi_1[\gamma; n]^{-1}, & D[\gamma; n] &= \overline{\Pi_1[\gamma; n]}^{-1} \end{aligned}$$

for all integers  $n$ , we derive from (1.8) the identity

$$A[\gamma; n]A[\gamma; n]^* = \text{Id}[n] - A[\gamma; n]\mathfrak{S}_1[\gamma^{-1}; n]A[\gamma; n]^* - C[\gamma; 1-n]^T \overline{C[\gamma; 1-n]}.$$

To conclude that  $\log \det A[n]A[n]^*$  is well-defined, we show by using variation techniques that  $C[\gamma; 1-n]^T \overline{C[\gamma; 1-n]}$  is trace class.

After the tedious effort spent on proving that the function  $\mathfrak{F}_n$  is well defined, we proceed to compute its derivative. We show that

$$\partial \mathfrak{F}_n(v) = \frac{6n^2 - 6n + 1}{12\pi i} \oint_{S^1} \mathcal{S}(g)(z)v(z)dz.$$

From this we conclude the *universal index theorem* on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  :

$$\det N_n = \det K[n]K[n]^* = \exp \left( -\frac{6n^2 - 6n + 1}{12\pi} S \right)$$

and

$$\partial \bar{\partial} \mathfrak{F}_n = -\frac{6n^2 - 6n + 1}{6\pi i} \omega_{WP},$$

which are the generalizations of (1.6) and (1.7) to our homogeneous space  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ . It also follows that

$$\det N_n = (\det N_1)^{6n^2 - 6n + 1},$$

a universal version of Mumford isomorphism [Mum77].

## 2. THE HOMOGENEOUS SPACES $S^1 \setminus \text{Diff}_+(S^1)$ AND $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$

In this section, we briefly recall some facts we need about the homogeneous spaces  $S^1 \setminus \text{Diff}_+(S^1)$  and  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ .

Let  $\mathbb{D}$  and  $\mathbb{D}^*$  be the unit circle and its exterior, and let  $S^1$  be the unit circle. Denote by  $\text{Diff}_+(S^1)$  the space of orientation preserving diffeomorphisms on the unit circle  $S^1$ . Under composition of mappings,  $\text{Diff}_+(S^1)$  is a Fréchet Lie group. We identify the subgroup of rotations with  $S^1$  itself. It defines a left action on  $\text{Diff}_+(S^1)$  and the resulting homogeneous space  $S^1 \setminus \text{Diff}_+(S^1)$  is a complex Kähler manifold, which is an object of much interest in string theory. According to Kirillov [Kir87] (see also [Teo04]), for every point  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ , identified with an element of  $\text{Diff}_+(S^1)$  fixing the point 1, there exists a unique conformal welding  $\gamma = g^{-1} \circ f$ , where  $f$  and  $g$  can be extended to diffeomorphisms on  $\hat{\mathbb{C}}$  in such a way that

**W1**  $f$  is holomorphic on  $\mathbb{D}$  and  $g$  is holomorphic on  $\mathbb{D}^*$ .

**W2**  $f(0) = 0, g(\infty) = \infty, f'(0) = 1$ .

We call  $(f, g)$  the pair of univalent functions associated to  $\gamma$ . The domains  $\Omega = f(\mathbb{D})$  and  $\Omega^* = g(\mathbb{D}^*)$  are simply connected domains in  $\hat{\mathbb{C}}$  with common boundary  $\mathcal{C} = f(S^1) = g(S^1)$  a  $C^\infty$  curve.

Under the inversion  $\mathfrak{I} : S^1 \setminus \text{Diff}_+(S^1) \rightarrow S^1 \setminus \text{Diff}_+(S^1)$ ,  $\mathfrak{I}(\gamma) = \gamma^{-1}$ , the pair of univalent functions  $(f[\gamma^{-1}], g[\gamma^{-1}])$  associated to  $\gamma^{-1}$  is related to the pair of univalent functions  $(f[\gamma], g[\gamma])$  associated to  $\gamma$  by

$$(2.1) \quad \begin{aligned} f[\gamma^{-1}](z) &= (\bar{r} \circ \iota \circ g[\gamma] \circ \iota)(z) = \bar{r} / \overline{g[\gamma](1/\bar{z})}, \\ g[\gamma^{-1}](z) &= (\bar{r} \circ \iota \circ f[\gamma] \circ \iota)(z) = \bar{r} / \overline{f[\gamma](1/\bar{z})}, \end{aligned}$$

where  $\iota : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is the inversion  $z \mapsto 1/\bar{z}$  on  $\hat{\mathbb{C}}$  and  $r = g'(\infty)$ .

Let  $\text{Möb}(S^1) = \text{PSU}(1, 1)$  be the group of Möbius transformations on the unit circle. The homogeneous space  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  is a quotient space of  $S^1 \setminus \text{Diff}_+(S^1)$ . We represent a point  $\gamma \in \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  with an element  $\gamma \in \text{Diff}_+(S^1)$  that fixes  $-1, -i$ , and  $1$ . Fix a point  $\gamma_0 \in \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , the points in the fiber of



$S^1 \setminus \text{Diff}_+(S^1) \rightarrow \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  over  $\gamma_0$  are given by  $\sigma_w \circ \gamma_0 \in S^1 \setminus \text{Diff}_+(S^1)$ ,  $w \in \mathbb{D}^*$ , where  $\sigma_w$  is the linear fractional transformation

$$\sigma_w(z) = \frac{1-w}{1-\bar{w}} \frac{1-z\bar{w}}{z-w}.$$

The tangent space at the origin of  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  consist of smooth vector fields

$$u(z) = \sum_{k \in \mathbb{Z} \setminus \{-1, 0, 1\}} c_k z^{k+1}, \quad c_{-k} = -\bar{c}_k, \quad z \in S^1$$

<sup>2</sup> on  $S^1$ . The holomorphic and anti-holomorphic tangent vectors corresponding to  $u$  are given respectively by

$$(2.2) \quad v = \sum_{k=2}^{\infty} c_k z^{k+1}, \quad \bar{v}(z) = \overline{v(z)} = \sum_{k=2}^{\infty} \bar{c}_k \bar{z}^{k+1}.$$

The tangent space at any other point  $\gamma \in \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  can be identified with the tangent space at the origin via right translation. To be more precise, let  $\gamma_t$  be a smooth curve in  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  such that  $\gamma_0 = \gamma$ . Define  $u_t = \gamma_t \circ \gamma^{-1}$  and

$$\dot{u}(z) = \left. \frac{d}{dt} \right|_{t=0} u_t(z) = \sum_{k \in \mathbb{Z}} c_k z^{k+1}.$$

It defines the holomorphic and anti-holomorphic tangent vectors  $v$  and  $\bar{v}$  with formulas given by (2.2).

The Weil Petersson metric is up to constants, the unique right-invariant Kähler metric on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ . At the tangent space of the origin, it is given by

$$\|v\|_{WP}^2 = 2\pi \sum_{k=2}^{\infty} (k^3 - k) |c_k|^2,$$

where  $v(z) = \sum_{k=2}^{\infty} c_k z^{k+1}$  is a holomorphic tangent vector.

### 3. SPACES AND FAMILIES OF HOLOMORPHIC DIFFERENTIALS

Given the simply connected domains  $\Omega$  and  $\Omega^*$ , and an integer  $n$ , a holomorphic  $n$  differential on  $\Omega$  (resp.  $\Omega^*$ ) is a holomorphic function  $\phi$  on  $\Omega$  (resp.  $\Omega^*$ ) such that

$$\phi(z) = O(1) \quad \text{as } z \rightarrow 0 \quad (\text{resp. } \phi(z) = O(z^{-2n}) \quad \text{as } z \rightarrow \infty).$$

For  $E = \Omega$  or  $\Omega^*$ , we denote by  $\mathcal{H}^n(E)$  the space of holomorphic  $n$ -differentials on  $E$ . For  $n \geq 1$ , we let  $\mathcal{H}_0^{1-n}(E)$  to be the subspace of  $\mathcal{H}^{1-n}(E)$  consisting of all  $\phi$  that satisfies

$$\begin{aligned} \phi(z) &= O(z^{2n-1}) \quad \text{as } z \rightarrow 0, \text{ if } E = \Omega \\ (\text{resp. } \phi(z) &= O(z^{-1}) \quad \text{as } z \rightarrow \infty, \text{ if } E = \Omega^*). \end{aligned}$$

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<sup>2</sup>Here we purposely differ from the usual convention by a factor of  $iz$ .

For  $n \geq 1$ , the Hilbert space  $A_{n,2}(E)$  is defined to be the subspace of  $\mathcal{H}^n(E)$  given by

$$A_{n,2}(E) = \left\{ \phi \text{ holomorphic on } E : \|\phi\|_{n,2}^2 = \iint_{\Omega} |\phi|^2 \rho_E^{1-n} < \infty \right\}.$$

Here  $\rho_E$  is the hyperbolic metric (i.e., metric with constant curvature  $-1$ ) density on  $E$ .

When  $n \geq 1$ , let

$$(3.1) \quad \alpha_n = \frac{2^{2n-2}}{(2n-2)!\pi}, \quad \beta_n = \frac{2^{2n-2}(2n-1)}{\pi} = (2n-1)!\alpha_n,$$

$$(3.2) \quad c[n]_k = \begin{cases} \sqrt{\alpha_n} \sqrt{\frac{(|k|+n-1)!}{(|k|-n)!}}, & \text{if } |k| \geq n, \\ \sqrt{\alpha_n}, & \text{if } |k| < n. \end{cases};$$

$$(3.3) \quad c[1-n]_k = \frac{\alpha_n}{c[n]_k} = \begin{cases} \sqrt{\alpha_n} \sqrt{\frac{(|k|-n)!}{(|k|+n-1)!}}, & \text{if } |k| \geq n, \\ \sqrt{\alpha_n}, & \text{if } |k| < n. \end{cases}$$

It is easy to check that

$$\{e^+[n]_k(z) = c[n]_k z^{k-n} : k \geq n\} \quad \text{and} \quad \{e^-[n]_k(z) = c[n]_k z^{-k-n} : k \geq n\}$$

are orthonormal bases for  $A_{n,2}(\mathbb{D})$  and  $A_{n,2}(\mathbb{D}^*)$  respectively. Since

$$\frac{d^{2n-1}}{dz^{2n-1}} (c[1-n]_k z^{\pm k+n-1}) = \begin{cases} (\operatorname{sgn}_n k) c[n]_k z^{\pm k-n}, & \text{if } |k| \geq n, \\ 0, & \text{if } |k| < n, \end{cases}$$

where

$$\operatorname{sgn}_n k = \begin{cases} 1, & \text{if } k \geq n, \\ -1, & \text{if } k < n, \end{cases}$$

we define

$$\begin{aligned} \{e^+[1-n]_k(z) = c[1-n]_k z^{k+n-1} : k \geq n\} & \quad \text{and} \\ \{e^-[1-n]_k(z) = c[1-n]_k z^{-k+n-1} : k \geq n\} \end{aligned}$$

as the corresponding bases of  $\mathcal{H}_0^{1-n}(\mathbb{D})$  and  $\mathcal{H}_0^{1-n}(\mathbb{D}^*)$ .

Given a point  $\gamma \in S^1 \setminus \operatorname{Diff}_+(S^1)$  with associated pair of univalent functions  $(f, g)$  and domains  $(\Omega, \Omega^*)$ , our goal is to define some canonical bases of  $\mathcal{H}_0^{1-n}(\Omega)$ ,  $\mathcal{H}_0^{1-n}(\Omega^*)$ ,  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$  for any integer  $n \geq 1$ .

Given a power series  $P(w) = \sum_{k \in \mathbb{Z}} c_k w^k$  and a subset  $S$  of integers, we define

$$P(w)_S = \sum_{k \in S} c_k w^k.$$

**3.1. Bases of  $\mathcal{H}_0^{1-n}(\Omega)$  and  $\mathcal{H}_0^{1-n}(\Omega^*)$ .** For  $k \geq n$ , define the Faber-type polynomials

$$\begin{aligned} u[1-n]_k(w) &= c[1-n]_k \left( (g^{-1}(w))^{k+n-1} (g^{-1})'(w)^{1-n} \right)_{\geq 2n-1}, \\ v[1-n]_k(w) &= c[1-n]_k \left( (f^{-1}(w))^{-k+n-1} (f^{-1})'(w)^{1-n} \right)_{\leq -1}. \end{aligned}$$

It is easy to see that  $u[1-n]_k$  is a polynomial of degree  $k+n-1$  in  $w$  and  $v[1-n]_k$  is a polynomial of degree  $k-n+1$  in  $w^{-1}$ . Moreover, by definition, they belong to  $\mathcal{H}_0^{1-n}(\Omega)$  and  $\mathcal{H}_0^{1-n}(\Omega^*)$  respectively. Therefore,  $\{u[1-n]_k : k \geq n\}$  is a basis for  $\mathcal{H}_0^{1-n}(\Omega)$  and  $\{v[1-n]_k : k \geq n\}$  is a basis for  $\mathcal{H}_0^{1-n}(\Omega^*)$ . For  $n=1$ ,  $u[0]_k(w)$  and  $v[0]_k(w)$  are up to normalization and the constant terms, the Faber polynomials (see e.g. [Pom75, Dur83, Teo03]) of the pair  $(f, g)$ . One can check by residue calculus that  $u[1-n]_k$  and  $v[1-n]_k$  are encoded in the expansion

$$(3.4) \quad \begin{aligned} \frac{g'(z)^n w^{2n-1}}{(g(z)-w)g(z)^{2n-1}} &= \frac{1}{\alpha_n} \sum_{k=n}^{\infty} u[1-n]_k(w) c[n]_k z^{-k-n}, \\ \frac{f'(z)^n}{f(z)-w} &= -\frac{1}{\alpha_n} \sum_{k=n}^{\infty} v[1-n]_k(w) c[n]_k z^{k-n}. \end{aligned}$$

Consider the expansions of  $u[1-n]_k \circ f(f')^{1-n}$ ,  $u[1-n]_k \circ g(g')^{1-n}$ ,  $v[1-n]_k \circ f(f')^{1-n}$ ,  $v[1-n]_k \circ g(g')^{1-n}$  around 0 or  $\infty$ :

$$(3.5) \quad \begin{aligned} u[1-n]_k(f(z)) f'(z)^{1-n} &= \sum_{l=n}^{\infty} A[1-n]_{lk} c[1-n]_l z^{l+n-1}, \\ u[1-n]_k(g(z)) g'(z)^{1-n} &= c[1-n]_k z^{k+n-1} + \sum_{l=1-n}^{\infty} B[1-n]_{lk} c[1-n]_l z^{-l+n-1}, \\ v[1-n]_k(f(z)) f'(z)^{1-n} &= c[1-n]_k z^{-k+n-1} + \sum_{l=1-n}^{\infty} C[1-n]_{lk} c[1-n]_l z^{l+n-1}, \\ v[1-n]_k(g(z)) g'(z)^{1-n} &= \sum_{l=n}^{\infty} D[1-n]_{lk} c[1-n]_l z^{-l+n-1}. \end{aligned}$$

Then

$$(3.6) \quad \begin{aligned} \frac{g'(z)^n f'(w)^{1-n} f(w)^{2n-1}}{(g(z)-f(w))g(z)^{2n-1}} &= \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} A[1-n]_{lk} c[1-n]_l c[n]_k w^{l+n-1} z^{-k-n}, \\ \frac{g'(z)^n g'(w)^{1-n} g(w)^{2n-1}}{(g(z)-g(w))g(z)^{2n-1}} &= \frac{1}{z-w} + \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=1-n}^{\infty} B[1-n]_{lk} c[1-n]_l c[n]_k w^{-l+n-1} z^{-k-n}, \\ \frac{f'(z)^n f'(w)^{1-n}}{f(z)-f(w)} &= \frac{1}{z-w} - \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=1-n}^{\infty} C[1-n]_{lk} c[1-n]_l c[n]_k w^{l+n-1} z^{k-n}, \\ \frac{f'(z)^n g'(w)^{1-n}}{f(z)-g(w)} &= -\frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} D[1-n]_{lk} c[1-n]_l c[n]_k w^{-l+n-1} z^{k-n}. \end{aligned}$$

We define the matrices  $A[1-n], \hat{B}[1-n], B[1-n], \hat{C}[1-n], C[1-n], D[1-n]$  by

$$\begin{aligned} A[1-n] &= (A[1-n]_{lk})_{l,k \geq n}, & D[1-n] &= (D[1-n]_{lk})_{l,k \geq n}, \\ B[1-n] &= (B[1-n]_{lk})_{l,k \geq n}, & C[1-n] &= (C[1-n]_{lk})_{l,k \geq n}, \\ \hat{B}[1-n] &= (B[1-n]_{l,k})_{l \geq 1-n, k \geq n}, & \hat{C}[1-n] &= (C[1-n]_{l,k})_{l \geq 1-n, k \geq n}. \end{aligned}$$

Using (2.1), it is straightforward to verify from (3.6) that under the inversion  $\gamma \rightarrow \gamma^{-1}$  on  $S^1 \setminus \text{Diff}_+(S^1)$ ,

(3.7)

$$A[\gamma^{-1}; 1-n] = \overline{D[\gamma; 1-n]}, \quad \hat{B}[\gamma^{-1}; 1-n] = \overline{\hat{C}[\gamma; 1-n]}, \quad B[\gamma^{-1}; 1-n] = \overline{C[\gamma; 1-n]}$$

for all  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ .

**3.2. Bases of  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$ .** There are two natural ways to choose bases for  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$ .

**First Choice.** For  $k \geq n$ , define the Faber-type polynomials

$$\begin{aligned} U[n]_k(w) &= c[n]_k ((g^{-1}(w))^{k-n} (g^{-1})'(w)^n)_{\geq 0}, \\ V[n]_k(w) &= c[n]_k ((f^{-1}(w))^{-k-n} (f^{-1})'(w)^n)_{\leq -2n}. \end{aligned}$$

It is easy to see that  $U[n]_k$  is a polynomial of degree  $k-n$  in  $w$ ,  $V[n]_k(w)$  is a polynomial of degree  $k+n$  in  $w^{-1}$  and  $V[n]_k(w) = O(w^{-2n})$  as  $w \rightarrow \infty$ . Therefore,  $\{U[n]_k(w) : k \geq n\}$  and  $\{V[n]_k(w) : k \geq n\}$  are bases of  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$  respectively. By residue calculus, it is easy to verify that the families  $\{U[n]_k(w)\}$  and  $\{V[n]_k(w)\}$  can be compactly defined by

$$\begin{aligned} (3.8) \quad \frac{g'(z)^{1-n}}{g(z)-w} &= \frac{1}{\alpha_n} \sum_{k=n}^{\infty} U[n]_k(w) c[1-n]_k z^{-k+n-1}, \\ \frac{f'(z)^{1-n} f(z)^{2n-1}}{(f(z)-w) w^{2n-1}} &= -\frac{1}{\alpha_n} \sum_{k=n}^{\infty} V[n]_k(w) c[1-n]_k z^{k+n-1}. \end{aligned}$$

From the definition of  $U[n]_k$  and  $V[n]_k$ , it is easy to see that  $U[n]_k \circ f(f')^n$ ,  $U[n]_k \circ g(g')^n$ ,  $V[n]_k \circ f(f')^n$  and  $V[n]_k \circ g(g')^n$  have expansions around 0 or  $\infty$  of the following form:

$$\begin{aligned} (3.9) \quad U[n]_k(f(z)) f'(z)^n &= \sum_{l=n}^{\infty} A[n]_{lk} c[n]_l z^{l-n}, \\ U[n]_k(g(z)) g'(z)^n &= c[n]_k z^{k-n} + \sum_{l=1-n}^{\infty} B[n]_{lk} c[n]_l z^{-l-n}, \\ V[n]_k(f(z)) f'(z)^n &= c[n]_k z^{-k-n} + \sum_{l=1-n}^{\infty} C[n]_{lk} c[n]_l z^{l-n}, \\ V[n]_k(g(z)) g'(z)^n &= \sum_{l=n}^{\infty} D[n]_{lk} c[n]_l z^{-l-n}. \end{aligned}$$

It follows that

(3.10)

$$\begin{aligned} \frac{g'(z)^{1-n} f'(w)^n}{g(z) - f(w)} &= \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} A[n]_{lk} c[n]_l c[1-n]_k w^{l-n} z^{-k+n-1}, \\ \frac{g'(z)^{1-n} g'(w)^n}{g(z) - g(w)} &= \frac{1}{z-w} + \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=1-n}^{\infty} B[n]_{lk} c[n]_l c[1-n]_k w^{l-n} z^{-k+n-1}, \\ \frac{f'(z)^{1-n} f'(w)^n f(z)^{2n-1}}{(f(z) - f(w)) f(w)^{2n-1}} &= \frac{1}{z-w} - \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=1-n}^{\infty} C[n]_{lk} c[n]_l c[1-n]_k w^{l-n} z^{k+n-1}, \\ \frac{f'(z)^{1-n} g'(w)^n f(z)^{2n-1}}{(f(z) - g(w)) g(w)^{2n-1}} &= -\frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} D[n]_{lk} c[n]_l c[1-n]_k w^{-l-n} z^{k+n-1}. \end{aligned}$$

We define the matrices  $A[n], B[n], \hat{B}[n], C[n], \hat{C}[n], D[n]$  by

$$\begin{aligned} A[n] &= (A[n]_{lk})_{l,k \geq n}, & D[n] &= (D[n]_{lk})_{l \geq n, k \geq n}, \\ B[n] &= (B[n]_{lk})_{l,k \geq n}, & C[n] &= (C[n]_{lk})_{l \geq n, k \geq n}, \\ \hat{B}[n] &= (B[n]_{lk})_{l \geq 1-n, k \geq n}, & \hat{C}[n] &= (C[n]_{lk})_{l \geq 1-n, k \geq n}. \end{aligned}$$

Under the inversion  $\gamma \rightarrow \gamma^{-1}$  on  $S^1 \setminus \text{Diff}_+(S^1)$ ,

$$A[\gamma^{-1}; n] = \overline{D[\gamma; n]}, \quad B[\gamma^{-1}; n] = \overline{C[\gamma; n]}, \quad \hat{B}[\gamma^{-1}; n] = \overline{\hat{C}[\gamma; n]}$$

for all  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ .

Comparing (3.6) and (3.10) gives the following relation.

$$(3.11) \quad A[1-n] = D[n]^T, \quad D[1-n] = A[n]^T.$$

**Second Choice.** Another natural bases  $\{p[n]_k : k \geq n\}$  and  $\{q[n]_k : k \geq n\}$  of  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$  are defined by

$$(3.12) \quad p[n]_k(w) = u[1-n]_k^{(2n-1)}(w), \quad q[n]_k(w) = -v[1-n]_k^{(2n-1)}(w).$$

It is easy to see that  $p[n]_k(w)$  is a polynomial of degree  $k-n$  in  $w$ ,  $q[n]_k(w)$  is a polynomial of degree  $k+n$  in  $w^{-1}$  and  $q[n]_k(w) = O(w^{-2n})$  as  $w \rightarrow \infty$ . Therefore,  $\{p[n]_k \mid k \geq n\}$  and  $\{q[n]_k \mid k \geq n\}$  indeed form bases of  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$  respectively. The expansions of  $p[n]_k \circ f(f')^n, p[n]_k \circ g(g')^n, q[n]_k \circ f(f')^n, q[n]_k \circ g(g')^n$  around 0 or  $\infty$  have the following form:

$$\begin{aligned} (3.13) \quad p[n]_k(f(z)) f'(z)^n &= \sum_{l=n}^{\infty} \mathfrak{A}[n]_{lk} c[n]_l z^{l-n}, \\ p[n]_k(g(z)) g'(z)^n &= \sum_{l=n}^k \mathfrak{P}[n]_{lk} c[n]_l z^{l-n} + \sum_{l=1-n}^{\infty} \mathfrak{B}[n]_{lk} c[n]_l z^{-l-n}, \\ q[n]_k(f(z)) f'(z)^n &= \sum_{l=n}^k \mathfrak{M}[n]_{lk} c[n]_l z^{-l-n} + \sum_{l=1-n}^{\infty} \mathfrak{C}[n]_{lk} c[n]_l z^{l-n}, \\ q[n]_k(g(z)) g'(z)^n &= \sum_{l=n}^{\infty} \mathfrak{D}[n]_{lk} c[n]_l z^{-l-n}. \end{aligned}$$

The matrices  $\mathfrak{A}[n], \mathfrak{B}[n], \hat{\mathfrak{B}}[n], \mathfrak{C}[n], \hat{\mathfrak{C}}[n], \mathfrak{D}[n], \mathfrak{P}[n], \mathfrak{M}[n]$  are defined respectively by

$$\begin{aligned}\mathfrak{A}[n] &= (\mathfrak{A}[n]_{lk})_{l,k \geq n}, & \mathfrak{D}[n] &= (\mathfrak{D}[n]_{lk})_{l,k \geq n}, \\ \mathfrak{B}[n] &= (\mathfrak{B}[n]_{lk})_{l,k \geq n}, & \mathfrak{C}[n] &= (\mathfrak{C}[n]_{lk})_{l,k \geq n}, \\ \hat{\mathfrak{B}}[n] &= (\mathfrak{B}[n]_{lk})_{l \geq 1-n, k \geq n}, & \hat{\mathfrak{C}}[n] &= (\mathfrak{C}[n]_{lk})_{l \geq 1-n, k \geq n}, \\ \mathfrak{P}[n] &= (\mathfrak{P}[n]_{lk})_{l,k \geq n}, & \mathfrak{M}[n] &= (\mathfrak{M}[n]_{lk})_{l,k \geq n}.\end{aligned}$$

Differentiating (3.4) with respect to  $w$   $(2n-1)$  times and using the definition (3.1) of  $\beta_n$ , we have

(3.14)

$$\begin{aligned}\beta_n \frac{g'(z)^n}{(g(z) - w)^{2n}} &= \sum_{k=n}^{\infty} u[1-n]_k^{(2n-1)}(w) c[n]_k z^{-k-n} = \sum_{k=n}^{\infty} p[n]_k(w) c[n]_k z^{-k-n}, \\ \beta_n \frac{f'(z)^n}{(f(z) - w)^{2n}} &= - \sum_{k=n}^{\infty} v[1-n]_k^{(2n-1)}(w) c[n]_k z^{-k-n} = \sum_{k=n}^{\infty} q[n]_k(w) c[n]_k z^{-k-n}.\end{aligned}$$

Therefore,

$$\begin{aligned}\beta_n \frac{g'(z)^n f'(w)^n}{(g(z) - f(w))^{2n}} &= \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} \mathfrak{A}[n]_{lk} c[n]_l c[n]_k w^{l-n} z^{-k-n}, \\ &= \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} \mathfrak{D}[n]_{lk} c[n]_l c[n]_k z^{-l-n} w^{k-n}.\end{aligned}$$

It follows that,

$$\mathfrak{A}[n] = \mathfrak{D}[n]^T.$$

Under the inversion  $\gamma \mapsto \gamma^{-1}$ , we have

$$\begin{aligned}\mathfrak{A}[\gamma^{-1}; n] &= \overline{\mathfrak{D}[\gamma; n]}, & \mathfrak{P}[\gamma^{-1}; n] &= \overline{\mathfrak{M}[\gamma; n]}, \\ \mathfrak{B}[\gamma^{-1}; n] &= \overline{\mathfrak{C}[\gamma; n]}, & \hat{\mathfrak{B}}[\gamma^{-1}; n] &= \overline{\hat{\mathfrak{C}}[\gamma; n]}.\end{aligned}$$

for all  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ .

From the definitions of  $p[n]_k$  and  $U[n]_k$ , we can write  $p[n]_k$  as a linear combination of  $U[n]_l$  with  $n \leq l \leq k$ . In fact, from the expansion of  $p[n]_k(g(z))g'(z)^n$  and  $U[n]_k(g(z))g'(z)^n$ , it is easy to conclude that

$$(3.15) \quad p[n]_k = \sum_{l=n}^k \mathfrak{P}[n]_{lk} U[n]_l.$$

Similarly, we have

$$q[n]_k = \sum_{l=n}^{\infty} \mathfrak{M}[n]_{lk} V[n]_l.$$

Therefore,  $\mathfrak{P}[n]$  and  $\mathfrak{M}[n]$  are transition matrices between the bases  $\{p[n]_k\}, \{U[n]_k\}$  and the bases  $\{q[n]_k\}, \{V[n]_k\}$  respectively. These give

$$(3.16) \quad \begin{aligned}\mathfrak{A}[n] &= A[n]\mathfrak{P}[n], & \mathfrak{B}[n] &= B[n]\mathfrak{P}[n], & \hat{\mathfrak{B}}[n] &= \hat{B}[n]\mathfrak{P}[n], \\ \mathfrak{D}[n] &= D[n]\mathfrak{M}[n], & \mathfrak{C}[n] &= C[n]\mathfrak{M}[n], & \hat{\mathfrak{C}}[n] &= \hat{C}[n]\mathfrak{M}[n].\end{aligned}$$

In case  $n = 1$ , differentiating (3.8) with respect to  $z$  and compare to (3.14), we find that  $p[1]_k = U[1]_k$  and  $q[1]_k = V[1]_k$  for all  $k \geq 1$ . Therefore,  $\mathfrak{P}[1] = \mathfrak{M}[1] = \text{Id}$  and

$$\mathfrak{A}[1] = A[1], \quad \mathfrak{B}[1] = B[1], \quad \mathfrak{C}[1] = C[1], \quad \mathfrak{D}[1] = D[1].$$

**3.3. A basis of  $\mathcal{H}^{1-n}(\Omega)$ .** For the purpose of proving our main theorem in a later section, we also need to consider the family of polynomials defined by

$$U[1-n]_k(w) = c[1-n]_k (g^{-1}(w)^{k+n-1} (g^{-1})'(w))_{\geq 0}, \quad k \geq 1-n.$$

Since  $U[1-n]_k(w)$  is a polynomial of degree  $k+n-1$  in  $w$ ,  $\{U[1-n]_k : k \geq 1-n\}$  form a basis of  $\mathcal{H}^{1-n}(\Omega)$ . By residue calculus,

$$\frac{g'(z)^n}{g(z) - w} = \frac{1}{\alpha_n} \sum_{k=1-n}^{\infty} U[1-n]_k(w) c[n]_k z^{-k-n}.$$

Let

$$\begin{aligned} U[1-n]_k(f(z)) f'(z)^{1-n} &= \sum_{l=1-n}^{\infty} \mathbb{A}[1-n]_{lk} c[1-n]_l z^{l+n-1}, \\ U[1-n]_k(g(z)) g'(z)^{1-n} &= c[1-n]_k z^{k+n-1} + \sum_{l=n}^{\infty} \mathbb{B}[1-n]_{lk} c[1-n]_l z^{-l+n-1}, \end{aligned}$$

and define the matrix

$$\mathbb{A}[1-n] = (\mathbb{A}[1-n]_{lk})_{l,k \geq 1-n}.$$

Then,

$$(3.17) \quad \frac{g'(z)^n f'(w)^{1-n}}{g(z) - f(w)} = \frac{1}{\alpha_n} \sum_{k=1-n}^{\infty} \sum_{l=1-n}^{\infty} \mathbb{A}[1-n]_{lk} c[n]_k c[1-n]_l w^{l+n-1} z^{-k-n}.$$

For  $k \geq n$ , we also define the holomorphic  $n$ -differentials  $\mathfrak{U}[n]_k$  by

$$\mathfrak{U}[n]_k(z) = \frac{1}{\alpha_n} \frac{d^{2n-1}}{dz^{2n-1}} (U[1-n]_k(f(z)) f'(z)^{1-n}) = \frac{1}{\alpha_n} \sum_{l=n}^{\infty} \mathbb{A}[1-n]_{lk} c[n]_l z^{l-n}.$$

Since  $f$  is a smooth function and  $U[1-n]_k(z)$  is a polynomial,  $\mathfrak{U}[n]_k \in A_{n,2}(\mathbb{D})$ .

#### 4. OPERATORS ON HILBERT SPACES

It is easy to verify that for any  $\phi \in A_{n,2}(\mathbb{D})$  or  $A_{n,2}(\mathbb{D}^*)$ , we have the following reproducing formula.

$$\phi(z) = \beta_n \iint_{\mathbb{D} \text{ or } \mathbb{D}^*} \frac{\phi(w) \rho(w)^{1-n}}{(1 - z\bar{w})^2} d^2 w.$$

Therefore, the kernel for the identity operator  $\text{id}[n]$  on  $A_{n,2}(\mathbb{D})$  and  $A_{n,2}(\mathbb{D}^*)$  are given by

$$\text{id}[n](z, w) = \frac{\beta_n}{(1 - z\bar{w})^{2n}} = \begin{cases} \sum_{k=n}^{\infty} c[n]_k^2 (z\bar{w})^{k-n}, & \text{if } z, w \in \mathbb{D}, \\ \sum_{k=n}^{\infty} c[n]_k^2 (z\bar{w})^{-k-n}, & \text{if } z, w \in \mathbb{D}^*. \end{cases}$$

In [Ber66], Bers defined the integral operator  $K[n]$  (resp.  $L[n]$ ) which maps anti-holomorphic  $n$ -differentials on  $\Omega^*$  (resp.  $\Omega$ ) to holomorphic  $n$ -differentials on  $\Omega$  (resp.  $\Omega^*$ ) by

$$\begin{aligned} (K[n]\bar{\phi})(z) &= \beta_n \iint_{\Omega^*} \frac{\overline{\phi(w)}\rho(w)^{1-n}}{(z-w)^{2n}} d^2w, \\ (L[n]\bar{\phi})(z) &= \beta_n \iint_{\Omega} \frac{\overline{\phi(w)}\rho(w)^{1-n}}{(z-w)^{2n}} d^2w. \end{aligned}$$

Here we want to show that  $K[n]$  maps  $\overline{A_{n,2}(\Omega^*)}$  into  $A_{n,2}(\Omega)$  and it is a bounded operator.

**Proposition 4.1.**  *$K[n]$  is a bounded integral operator mapping  $\overline{A_{n,2}(\Omega^*)}$  into  $A_{n,2}(\Omega)$ .*

*Proof.* We have to show that there is a constant  $M$  such that for all  $\phi \in A_{n,2}(\Omega^*)$

$$\|K\bar{\phi}\|_{n,2} \leq \|\phi\|_{n,2}.$$

The result is well-known for  $n = 1$  and in this case,  $M = 1$  will work (see e.g. [TT06]). Let  $\eta_1(w)$  (resp.  $\eta_2(z)$ ) to be the distance of  $w \in \Omega^*$  (resp.  $z \in \Omega$ ) to the boundary of  $\Omega^*$  (resp.  $\Omega$ ). Classical inequality says that (see e.g. [Nag88])

$$\frac{1}{4} \leq \eta_i(z)^2 \rho_i(z) \leq 4, \quad i = 1, 2.$$

Therefore, for any integer  $k \geq 1$ ,

$$\frac{\rho(w)^{1-k}}{|z-w|^{2k-2}} \leq \frac{\rho(w)^{1-k}}{\eta_1(w)^{2k-2}} \leq 4^{k-1}$$

and

$$\begin{aligned} \iint_{\Omega^*} \frac{\rho(w)^{1-k}}{|z-w|^{4k}} d^2w &\leq 4^{k-1} \iint_{|z-w| \geq \eta_2(z)} \frac{1}{|z-w|^{2k+2}} d^2w \\ &= 2^{2k-1} \pi \int_{\eta_2(z)}^{\infty} \frac{r dr}{r^{2k+2}} \\ &= \frac{2^{2k-2} \pi}{k} \eta_2(z)^{-2k} \leq \frac{2^{4k-2} \pi}{k} \rho(z)^k. \end{aligned}$$

This implies that for  $n \geq 2$ ,

(4.1)

$$\begin{aligned} \|K[n]\bar{\phi}\|_{n,2}^2 &= \beta_n^2 \iint_{\mathbb{D}} \left| \iint_{\mathbb{D}^*} \frac{\overline{\phi(w)}\rho(w)^{1-n}}{(z-w)^{2n}} d^2w \right|^2 \rho(z)^{1-n} d^2z \\ &\leq \beta_n^2 \iint_{\mathbb{D}} \left( \iint_{\mathbb{D}^*} \frac{\rho(w)^{2-n}}{|z-w|^{4n-4}} d^2w \right) \left( \iint_{\mathbb{D}^*} \frac{|\phi(w)|^2 \rho(w)^{-n}}{|z-w|^4} d^2w \right) \rho(z)^{1-n} d^2z \\ &\leq \frac{2^{4n-6} \pi}{n-1} \beta_n^2 \iint_{\mathbb{D}} \iint_{\mathbb{D}^*} \frac{|\phi(w)|^2 \rho(w)^{-n}}{|z-w|^4} d^2w d^2z. \end{aligned}$$



By our result in [TT06], for  $w \in \mathbb{D}^*$ ,

$$\frac{1}{\pi^2} \iint_{\mathbb{D}} \frac{d^2 z}{|z - w|^4} = L[1]L[1]^*(w, w) \leq \text{Id}[1](w, w) = \frac{1}{4\pi} \rho(w).$$

Therefore, (4.1) is bounded by

$$\frac{2^{4n-8}\pi^2}{n-1} \beta_n^2 \iint_{\mathbb{D}^*} |\phi(w)|^2 \rho(w)^{1-n} d^2 w = \frac{2^{4n-8}\pi^2}{n-1} \beta_n^2 \|\phi\|_{n,2}^2.$$

This implies the assertion.  $\square$

In this proposition, we show that the norm of the operator  $K[n]$  is less than or equal to  $(2^{2n-4}\pi/\sqrt{n-1})\beta_n$ . In fact, we conjecture that it is less than or equal to 1. We gave some justification of this conjecture in the Appendix.

Under the isomorphism  $A_{n,2}(\Omega) \simeq A_{n,2}(\mathbb{D})$  and  $A_{n,2}(\Omega^*) \simeq A_{n,2}(\mathbb{D}^*)$  induced by  $f$  and  $g$  respectively, we can consider  $K[n]$  (resp.  $L[n]$ ) as an operator from  $\overline{A_{n,2}(\mathbb{D}^*)}$  (resp.  $\overline{A_{n,2}(\mathbb{D})}$ ) to  $A_{n,2}(\mathbb{D})$  (resp.  $A_{n,2}(\mathbb{D}^*)$ ). It is easy to check that in this perspective, the kernel of  $K[n]$  and  $L[n]$  are given by

$$\mathfrak{A}[n](z, w) = \frac{f'(z)^n g'(w)^n}{(f(z) - g(w))^{2n}} \quad \text{and} \quad \mathfrak{D}[n](z, w) = \frac{g'(z)^n f'(w)^n}{(g(z) - f(w))^{2n}}.$$

Therefore,

**Lemma 4.2.** *The matrices  $\mathfrak{A}[n]$  and  $\mathfrak{D}[n]$  define bounded operators on  $\ell^2$ .*

For any integer  $n \geq 1$ , we define kernels

$$(4.2) \quad \begin{aligned} \mathcal{A}[n](z, w) &= \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} A[n]_{l,k} c[n]_l c[n]_k z^{-l-n} w^{k-n}, \\ \mathcal{C}[n](z, w) &= \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} C[1-n]_{l,k} c[n]_l c[n]_k w^{l-n} z^{k-n}. \end{aligned}$$

We are going to show in later sections that  $\mathcal{A}[n]$  and  $\mathcal{C}[n]$  define bounded operators on  $\ell^2$ . Therefore, they can be considered as integral operators  $\mathcal{A}[n] : \overline{A_{n,2}(\mathbb{D})} \rightarrow A_{n,2}(\mathbb{D}^*)$  and  $\mathcal{C}[n] : \overline{A_{n,2}(\mathbb{D})} \rightarrow A_{n,2}(\mathbb{D})$ . From (3.10) and (3.6), we find that

$$(4.3) \quad \begin{aligned} \mathcal{A}[n](z, w) &= -\alpha_n \frac{d^{2n-1}}{dz^{2n-1}} \left( \frac{g'(z)^{1-n} f'(w)^n}{g(z) - f(w)} \right), \\ \mathcal{C}[n](z, w) &= -\alpha_n \frac{d^{2n-1}}{dw^{2n-1}} \left( \frac{f'(z)^n f'(w)^{1-n}}{f(z) - f(w)} - \frac{1}{z - w} \right). \end{aligned}$$

Therefore,  $\mathcal{A}[n]$  and  $\mathcal{C}[n]^T$  can be considered as the composition of the integral operators  $\mathcal{A}_0[n]$ ,  $\mathcal{C}_0[n]$  mapping anti-holomorphic  $n$  differentials to holomorphic  $1-n$  differentials defined by

$$\begin{aligned} (\mathcal{A}_0[n]\bar{\phi})(z) &= -\alpha_n \iint_{\mathbb{D}} \frac{g'(z)^{1-n} f'(w)^n \overline{\phi(w)} \rho(w)^{1-n}}{g(z) - f(w)} d^2 w, \\ (\mathcal{C}_0[n]\bar{\phi})(z) &= \alpha_n \iint_{\mathbb{D}} \left( \frac{f'(z)^{1-n} f'(w)^n}{f(z) - f(w)} - \frac{1}{z - w} \right) \overline{\phi(w)} \rho(w)^{1-n} d^2 w, \end{aligned}$$

with the map  $d_n$  of taking  $2n-1$  times derivatives of a  $1-n$  differential.

Finally, we also define the operator  $\mathcal{D}[n] : \overline{A_{n,2}(\mathbb{D}^*)} \rightarrow A_{n,2}(\mathbb{D})$  by the kernel

$$(4.4) \quad \begin{aligned} \mathcal{D}[n](z, w) &= \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} \mathbb{A}[1-n]_{lk} c[n]_l c[n]_k w^{l-n} z^{-k-n} \\ &= \frac{d^{2n-1}}{dw^{2n-1}} \left( \frac{g'(z)^n f'(w)^{1-n}}{g(z) - f(w)} \right) - \sum_{k=1-n}^{n-1} \mathfrak{U}[n]_k(w) c[n]_k z^{-k-n}. \end{aligned}$$

## 5. PERIOD MATRICES OF HOLOMORPHIC $n$ -DIFFERENTIALS

For a domain  $E$ , the period matrix of holomorphic  $n$  differentials  $N_n(E)$  can be defined after we specify a choice of basis for  $A_{n,2}(E)$ . Since there are two natural choices of bases for  $A_{n,2}(\Omega)$  and  $A_{n,2}(\Omega^*)$  respectively, we can define  $N_n(\Omega)$  and  $N_n(\Omega^*)$  in two different ways.

**Definition 1.** We define the period matrices  $\mathcal{N}_n(\Omega)$  and  $\mathcal{N}_n(\Omega^*)$  by the bases  $\{U[n]_k(z) : k \geq n\}$  and  $\{V[n]_k(z) : k \geq n\}$ . More precisely,

$$\begin{aligned} \mathcal{N}_n(\Omega)_{lk} &= \langle U[n]_l, U[n]_k \rangle_{n,2} = \iint_{\Omega} U[n]_l(w) \overline{U[n]_k(w)} \rho_{\Omega}(w)^{1-n} d^2 w, \\ \mathcal{N}_n(\Omega^*)_{lk} &= \langle V[n]_l, V[n]_k \rangle_{n,2} = \iint_{\Omega^*} V[n]_l(w) \overline{V[n]_k(w)} \rho_{\Omega^*}(w)^{1-n} d^2 w. \end{aligned}$$

**Definition 2.** We define the period matrices  $N_n(\Omega)$  and  $N_n(\Omega^*)$  by the bases  $\{p[n]_k(z) : k \geq n\}$  and  $\{q[n]_k(z) : k \geq n\}$ . More precisely,

$$\begin{aligned} N_n(\Omega)_{lk} &= \langle p[n]_l, p[n]_k \rangle_{n,2} = \iint_{\Omega} p[n]_l(w) \overline{p[n]_k(w)} \rho_{\Omega}(w)^{1-n} d^2 w, \\ N_n(\Omega^*)_{lk} &= \langle q[n]_l, q[n]_k \rangle_{n,2} = \iint_{\Omega^*} q[n]_l(w) \overline{q[n]_k(w)} \rho_{\Omega^*}(w)^{1-n} d^2 w. \end{aligned}$$

Using the fact that  $\{e^+[n]_k \mid k \geq n\}$  is an orthonormal basis for  $A_{n,2}(\mathbb{D})$ , it is easy to compute that

$$\begin{aligned} \langle U[n]_l, U[n]_k \rangle_{n,2} &= \iint_{\Omega} U[n]_l(w) \overline{U[n]_k(w)} \rho_{\Omega}(w)^{1-n} d^2 w \\ &= \iint_{\mathbb{D}} U[n]_l(f(z)) f'(z)^n \overline{U[n]_k(f(z)) f'(z)^n} \rho_{\mathbb{D}}(z)^{1-n} d^2 z \\ &= \iint_{\mathbb{D}} \sum_{m=n}^{\infty} A[n]_{ml} c[n]_m z^{m-n} \sum_{j=n}^{\infty} \overline{A[n]_{jk} c[n]_j z^{j-n}} \rho_{\mathbb{D}}(z)^{1-n} d^2 z \\ &= \sum_{m=n}^{\infty} A[n]_{ml} \overline{A[n]_{mk}} \\ &= (A[n]^T \overline{A[n]})_{lk}. \end{aligned}$$

This gives

$$\mathcal{N}_n(\Omega) = A[n]^T \overline{A[n]} = D[1-n] D[1-n]^*.$$

Similarly, we find that

$$\begin{aligned}\mathcal{N}_n(\Omega^*) &= D[n]^T \overline{D[n]} = A[1-n]A[1-n]^*, \\ N_n(\Omega) &= \mathfrak{A}[n]^T \overline{\mathfrak{A}[n]} = \mathfrak{D}[n]\mathfrak{D}[n]^*, \\ N_n(\Omega^*) &= \mathfrak{D}[n]^T \overline{\mathfrak{D}[n]} = \mathfrak{A}[n]\mathfrak{A}[n]^*.\end{aligned}$$

## 6. THE TRANSITION MATRICES $\mathfrak{P}[n]$ AND $\mathfrak{M}[n]$

In this section, we are going to discuss the properties of the matrices  $\mathfrak{P}[n]$  and  $\mathfrak{M}[n]$  for any integer  $n \geq 1$ .

First, we define the Schwarzian derivative of a diffeomorphism  $T : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  by

$$\mathcal{S}(T) = \frac{T_{zzz}}{T_z} - \frac{3}{2} \left( \frac{T_{zz}}{T_z} \right)^2 = \left( \frac{T_{zz}}{T_z} \right)_z - \frac{1}{2} \left( \frac{T_{zz}}{T_z} \right)^2.$$

Here the subscript  $z$  denotes partial derivative with respect to  $z$ . We denote  $\mathfrak{R}_{\mathbb{C}}[T]$  the ring  $\mathbb{C}[\mathcal{S}(T), \mathcal{S}(T)_z, \mathcal{S}(T)_{zz}, \dots]$  whose elements are polynomials in  $\mathcal{S}(T)^{(n)}$ ,  $n \geq 0$ .<sup>3</sup> When  $T$  is a fractional linear transformation,  $\mathcal{S}(T) = 0$  and therefore  $\mathfrak{R}_{\mathbb{C}}[T] = \mathbb{C}$ .

For  $n = 1$ , we have seen that  $\mathfrak{P}[1] = \mathfrak{M}[1] = \text{Id}$ .

For  $n \geq 2$ , we have to apply the following well-known fact:

**Lemma 6.1.** *Let  $E$  be a domain on  $\mathbb{C}$ ,  $h : E \rightarrow \mathbb{C}$  a meromorphic function on  $E$  and  $T : \mathbb{C} \rightarrow \mathbb{C}$  a diffeomorphism. For any integer  $n \geq 2$ , there exists  $\xi[n]_k \in \mathfrak{R}_{\mathbb{C}}[T]$ ,  $2 \leq k \leq 2n-1$  such that*

$$\begin{aligned}(h \circ T(T_z)^{1-n})^{(2n-1)} &= h^{(2n-1)} \circ T(T_z)^n - \xi[n]_2 (h \circ T(T_z)^{1-n})^{(2n-3)} \\ &\quad - \xi[n]_3 (h \circ T(T_z)^{1-n})^{(2n-4)} - \dots - \xi[n]_{2n-1} (h \circ T(T_z)^{1-n}).\end{aligned}$$

For example, when  $n = 2$ , we have  $\xi[2]_2 = 2\mathcal{S}(T)$ ,  $\xi[2]_3 = \mathcal{S}(T)_z$ , or equivalently

$$\left( \frac{h \circ T}{T_z} \right)_{zzz} = h \circ T T_z^2 - 2\mathcal{S}(T) \left( \frac{h \circ T}{T_z} \right)_z - \mathcal{S}(T)_z \left( \frac{h \circ T}{T_z} \right).$$

When  $n = 3$ ,  $\xi[3]_2 = 10\mathcal{S}(T)$ ,  $\xi[3]_3 = 15\mathcal{S}(T)_z$ ,  $\xi[3]_4 = 9\mathcal{S}(T)_{zz} + 16\mathcal{S}(T)^2$ ,  $\xi[3]_5 = 2\mathcal{S}(T)_{zzz} + 16\mathcal{S}(T)\mathcal{S}(T)_z$ , or equivalently,

$$\begin{aligned}\left( \frac{h \circ T}{T_{zz}} \right)^{(5)} &= h \circ T T_z^3 - 10\mathcal{S}(T) \left( \frac{h \circ T}{T_{zz}} \right)_{zzz} - 15\mathcal{S}(T)_z \left( \frac{h \circ T}{T_{zz}} \right)_{zz} \\ &\quad - (9\mathcal{S}(T)_{zz} + 16\mathcal{S}(T)^2) \left( \frac{h \circ T}{T_{zz}} \right)_z - (2\mathcal{S}(T)_{zzz} + 16\mathcal{S}(T)\mathcal{S}(T)_z) \left( \frac{h \circ T}{T_{zz}} \right).\end{aligned}$$

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<sup>3</sup>Here for a function  $F$ ,  $F^{(n)}$  denotes the  $n$ -partial derivatives with respect to  $z$ .

Now, we differentiate the formulas in (3.8) with respect to  $z$   $(2n - 1)$  times:

$$\begin{aligned}
 (6.1) \quad \beta_n \frac{g'(z)^n}{(g(z) - w)^{2n}} &= \sum_{k=n}^{\infty} U[n]_k(w) c[n]_k z^{-k-n} \\
 &\quad + \xi[g; n]_2(z) \sum_{k=n}^{\infty} U[n]_k(w) c[1-n]_k \frac{(k+n-3)!}{(k-n)!} z^{-k-n+2} \\
 &\quad - \xi[g; n]_3(z) \sum_{k=n}^{\infty} U[n]_k(w) c[1-n]_k \frac{(k+n-4)!}{(k-n)!} z^{-k-n+3} \\
 &\quad + \dots - \xi[g; n]_{2n-1}(z) \sum_{k=n}^{\infty} U[n]_k(w) c[1-n]_k z^{-k+n-1}.
 \end{aligned}$$

It is easy to check that for  $2 \leq m \leq 2n - 1$ ,  $(-1)^m \xi[g; n]_m(z)$  has the expansion

$$(-1)^m \xi[g; n]_m(z) = \sum_{k=2}^{\infty} \Xi[g; n; m]_k c[2]_k z^{-k-m}.$$

For  $k < 2$ , we let  $\xi[g; n; m]_k = 0$ . It follows from (3.14), (3.15) and (6.1) that

$$(6.2) \quad \mathfrak{P}[n]_{lk} = \delta_{lk} + \frac{c[1-n]_l c[2]_{k-l}}{c[n]_k} \sum_{m=2}^{2n-1} \frac{(l+n-m-1)!}{(l-n)!} \Xi[g; n; m]_{k-l}.$$

Similarly, we have

$$(6.3) \quad \mathfrak{M}[n]_{lk} = \delta_{lk} + \frac{c[1-n]_l c[2]_{k-l}}{c[n]_k} \sum_{m=2}^{2n-1} \frac{(l+n-1)!}{(l-n+m)!} \Xi[f; n; m]_{k-l},$$

where

$$\xi[f; n]_m(z) = \sum_{k=m}^{\infty} \Xi[f; n; m]_k c[2]_k z^{k-m}.$$

and  $\Xi[f; n; m]_k = 0$  for  $k < m$ .

*Remark 6.2.* In the case  $n = 2$ , we let

$$\mathcal{S}(g)(z) = \sum_{k=2}^{\infty} g_k c[2]_k z^{-k-2}, \quad \mathcal{S}(f)(z) = \sum_{k=2}^{\infty} f_k c[2]_k z^{k-2}.$$

Then

$$\begin{aligned}
 \xi[g; 2]_2(z) &= 2\mathcal{S}(g)(z) = \sum_{k=2}^{\infty} (2g_k) c[2]_k z^{-k-2}, \\
 -\xi[g; 2]_3(z) &= -\mathcal{S}(g)'(z) = \sum_{k=2}^{\infty} ((k+2)g_k) c[2]_k z^{-k-3} \\
 \xi[f; 2]_2(z) &= 2\mathcal{S}(f)(z) = \sum_{k=2}^{\infty} (2f_k) c[2]_k z^{k-2}, \\
 \xi[f; 2]_3(z) &= \mathcal{S}(f)'(z) = \sum_{k=3}^{\infty} ((k-2)f_k) c[2]_k z^{k-3}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}\mathfrak{P}[2]_{lk} &= \delta_{lk} + \frac{c[-1]_l c[2]_{k-l}}{c[2]_k} (k+l) g_{k-l}, \\ \mathfrak{M}[2]_{lk} &= \delta_{lk} + \frac{c[-1]_l c[2]_{k-l}}{c[2]_k} (k+l) f_{k-l}.\end{aligned}$$

We see from (6.2) and (6.3) that the matrices  $\mathfrak{P}[n]$  and  $\mathfrak{M}[n]$  can be written as

$$\mathfrak{P}[n] = \text{Id} + \mathfrak{P}_0[n], \quad \mathfrak{M}[n] = \text{Id} + \mathfrak{M}_0[n],$$

where  $\mathfrak{P}_0[n]$  and  $\mathfrak{M}_0[n]$  are strictly upper triangular matrices. We can show more: they are in fact trace class operators.

**Lemma 6.3.** *Let  $K : A_{n,2}(\mathbb{D}) \rightarrow A_{n,2}(\mathbb{D})$  be an operator with kernel  $K(z, w)$ . Then*

**I.**  *$K$  is a Hilbert-Schmidt operator if and only if*

$$\iint_{\mathbb{D}} \iint_{\mathbb{D}} |K(z, w)|^2 \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w < \infty.$$

**II.** *Moreover, if*

$$\iint_{\mathbb{D}} \left( \iint_{\mathbb{D}} |K(z, w)|^2 \rho(w)^{1-n} d^2 w \right)^{1/2} \rho(z)^{1-(n/2)} d^2 z < \infty$$

or

$$\iint_{\mathbb{D}} \left( \iint_{\mathbb{D}} |K(z, w)|^2 \rho(z)^{1-n} d^2 z \right)^{1/2} \rho(w)^{1-(n/2)} d^2 w < \infty$$

then  $K$  is a trace class operator.

*Proof.* **I** is well known. If  $K$  is Hilbert-Schmidt, then  $K$  is compact. Therefore there exist  $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$  and an orthonormal basis  $\{\varphi_j \mid j \geq 1\}$  of  $A_{n,2}(\mathbb{D})$  such that

$$(KK^*)(z, w) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(z) \overline{\varphi_j(w)}.$$

Similarly, there exists an orthonormal basis  $\{\psi_j \mid j \geq 1\}$  of  $A_{n,2}(\mathbb{D})$  such that

$$(K^*K)(z, w) = \sum_{j=1}^{\infty} \lambda_j \psi_j(z) \overline{\psi_j(w)}.$$

$K$  is a trace class operator if and only if

$$\sum_{j=1}^{\infty} \lambda_j^{1/2} < \infty.$$

By Cauchy-Schwarz inequality,

$$(6.4) \quad \left( \sum_{j=1}^{\infty} \lambda_j^{1/2} \varphi_j(z) \overline{\varphi_j(z)} \right)^2 \leq \sum_{j=1}^{\infty} \lambda_j |\varphi_j(z)|^2 \sum_{j=1}^{\infty} |\varphi_j(z)|^2.$$

Since

$$\sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(w)}$$

is the kernel of the identity operator of  $A_{n,2}(\mathbb{D})$ , it is equal to  $\text{id}[n](z, w)$ . Therefore,

$$\sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(z)} = \frac{\beta_n}{(1 - z\bar{z})^{2n}} = \frac{\beta_n}{4^n} \rho(z)^n.$$

It follows from (6.4) that

$$\sum_{j=1}^{\infty} \lambda_j^{1/2} \varphi_j(z) \overline{\varphi_j(z)} \leq 2^{-n} \sqrt{\beta_n} \left( \sum_{j=1}^{\infty} \lambda_j |\varphi_j(z)|^2 \right)^{1/2} \rho(z)^{n/2}.$$

Consequently,

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j^{1/2} &= \iint_{\mathbb{D}} \sum_{j=1}^{\infty} \lambda_j^{1/2} \varphi_j(z) \overline{\varphi_j(z)} \rho(z)^{1-n} d^2 z \\ &\leq 2^{-n} \sqrt{\beta_n} \iint_{\mathbb{D}} \left( \sum_{j=1}^{\infty} \lambda_j |\varphi_j(z)|^2 \right)^{1/2} \rho(z)^{1-(n/2)} d^2 z \\ &= 2^{-n} \sqrt{\beta_n} \iint_{\mathbb{D}} \left( \iint_{\mathbb{D}} |K(z, w)|^2 \rho(w)^{1-n} d^2 w \right)^{1/2} \rho(z)^{1-(n/2)} d^2 z. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j^{1/2} &= \iint_{\mathbb{D}} \sum_{j=1}^{\infty} \lambda_j^{1/2} \psi_j(z) \overline{\psi_j(z)} \rho(z)^{1-n} d^2 z \\ &= 2^{-n} \sqrt{\beta_n} \iint_{\mathbb{D}} \left( \iint_{\mathbb{D}} |K(z, w)|^2 \rho(z)^{1-n} d^2 z \right)^{1/2} \rho(w)^{1-(n/2)} d^2 w. \end{aligned}$$

The assertion **II** of the lemma follows.  $\square$

**Proposition 6.4.** *For all  $2 \leq m \leq 2n - 1$ , the operator  $\mathfrak{M}_0[n]$  defines a trace class operator on  $\ell^2$ .*

*Proof.* For  $2 \leq m \leq 2n - 1$ , we define the matrices  $\mathfrak{M}_m[n]$  by

$$\mathfrak{M}_m[n]_{lk} = \frac{c[1-n]_l c[2]_{k-l}}{c[n]_k} \frac{(l+n-1)!}{(l-n+m)!} \Xi[f; n; m]_{k-l},$$

so that

$$\mathfrak{M}_0[n] = \sum_{m=2}^{2n-1} \mathfrak{M}_m[n].$$

It is sufficient to show that for all  $2 \leq m \leq 2n - 1$ ,  $\mathfrak{M}_m[n]$  defines a trace class operator. Denote by  $M_m[n] : A_{n,2}(\mathbb{D}) \rightarrow A_{n,2}(\mathbb{D})$  the corresponding operator, i.e.

the operator with kernel

$$\begin{aligned}
M_m[n](z, w) &= \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} \mathfrak{M}_m[n]_{lk} c[n]_l z^{l-n} c[n]_k \bar{w}^{k-n} \\
&= \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} c[1-n]_l c[2]_{k-l} c[n]_l \frac{(l+n-1)!}{(l-n+m)!} \Xi[f; n; m]_{k-l} z^{l-n} \bar{w}^{k-n} \\
&= \alpha_n \sum_{k=n+m}^{\infty} \sum_{l=n}^{k-m} c[2]_{k-l} \frac{(l+n-1)!}{(l-n+m)!} \Xi[f; n; m]_{k-l} z^{l-n} \bar{w}^{k-n} \\
&= \alpha_n \sum_{l=n}^{\infty} \sum_{k=l+m}^{\infty} c[2]_{k-l} \frac{(l+n-1)!}{(l-n+m)!} \Xi[f; n; m]_{k-l} z^{l-n} \bar{w}^{k-n} \\
&= \alpha_n \sum_{l=n}^{\infty} \sum_{k=m}^{\infty} c[2]_k \frac{(l+n-1)!}{(l-n+m)!} \Xi[f; n; m]_k z^{l-n} \bar{w}^{k+l-n} \\
&= \alpha_n \sum_{l=n}^{\infty} \frac{(l+n-1)!}{(l-n+m)!} \xi[f; n]_m(\bar{w}) z^{l-n} \bar{w}^{l-n+m}.
\end{aligned}$$

We compute

$$\begin{aligned}
&\iint_{\mathbb{D}} |M_m[n](z, w)|^2 \rho(z)^{1-n} d^2 z \\
&= \alpha_n^2 \sum_{l=n}^{\infty} \left| \frac{(l+n-1)!}{(l-n+m)!} \xi[f; n]_m(\bar{w}) \right|^2 |w|^{2l-2n+2m} \frac{1}{c[n]_l^2} \\
&= \alpha_n^2 |\xi[f; n]_m(\bar{w})|^2 \sum_{l=n}^{\infty} \frac{(l+n-1)!(l-n)!}{(l-n+m)!^2} |w|^{2l-2n+2m} \\
&= \alpha_n^2 |\xi[f; n]_m(\bar{w})|^2 \sum_{l=n}^{\infty} \frac{(l+n-m-1)!}{(l-n+m)!} |w|^{2l-2n+2m} \frac{(l+n-1)!(l-n)!}{(l+n-m-1)!(l-n+m)!}.
\end{aligned}$$

Since

$$\lim_{l \rightarrow \infty} \frac{(l+n-1)!(l-n)!}{(l+n-m-1)!(l-n+m)!} = \lim_{l \rightarrow \infty} \frac{(l+n-1)(l+n-2) \dots (l+n-m)}{(l-n+m)(l-n+m-1) \dots (l-n+1)} = 1,$$

There exists a constant  $B > 0$  such that

$$\frac{(l+n-1)!(l-n)!}{(l+n-m-1)!(l-n+m)!} \leq B$$

for all  $l \geq n$ . Therefore,

$$\begin{aligned}
&\iint_{\mathbb{D}} |M_m[n](z, w)|^2 \rho(z)^{1-n} d^2 z \\
&\leq \alpha_n B |\xi[f; n]_m(\bar{w})|^2 \sum_{l=n-m}^{\infty} \frac{(l+n-m-1)!}{(l-n+m)!} |w|^{2l-2n+2m} \\
&= \alpha_n B (2n-2m-1)! \frac{|\xi[f; n]_m(\bar{w})|^2}{(1-|w|^2)^{2n-2m}} \\
&= 2^{-2n+2m} \alpha_n B (2n-2m-1)! |\xi[f; n]_m(\bar{w})|^2 \rho(w)^{n-m}.
\end{aligned}$$

Consequently,

$$\begin{aligned} & \iint_{\mathbb{D}} \iint_{\mathbb{D}} |M_m[n](z, w)|^2 \rho(z)^{1-n} d^2 z \rho(w)^{1-n} d^2 w \\ & \leq 2^{-2n+2m} \alpha_n B(2n-2m-1)! \iint_{\mathbb{D}} |\xi[f; n]_m(\bar{w})|^2 \rho(w)^{1-m} d^2 w. \end{aligned}$$

Since  $f$  is smooth, so is  $\xi[f; n]_m(\bar{w})$ . Therefore, the last integral is finite and we conclude that  $M_m[n]$  is a Hilbert-Schmidt operator. On the other hand, by the same reasoning, the integral

$$\begin{aligned} & \iint_{\mathbb{D}} \left( \iint_{\mathbb{D}} |M_m[n](z, w)|^2 \rho(z)^{1-n} d^2 z \right)^{1/2} \rho(w)^{1-(n/2)} d^2 w \\ & \leq 2^{-n+m} \sqrt{\alpha_n B_2(2n-2m-1)!} \iint_{\mathbb{D}} |\xi[f; n]_m(\bar{w})| \rho(w)^{1-(m/2)} d^2 w \end{aligned}$$

is also finite. Therefore,  $M_m[n]$  is a trace class operator.  $\square$

Using the fact that  $\mathfrak{P}[\gamma; n] = \overline{\mathfrak{M}[\gamma^{-1}; n]}$ , we conclude that

**Corollary 6.5.** *For all  $n \geq 2$ ,  $\mathfrak{P}_0[n]$  defines a trace class operator on  $\ell^2$ .*

Now it follows from (3.11), (3.16) and Lemma 4.2 that

**Lemma 6.6.** *For all  $n \geq 1$ , the matrices  $A[\gamma; 1-n]$ ,  $D[\gamma; 1-n]$ ,  $A[n]$ ,  $D[n]$  define bounded operators on  $\ell^2$ .*

From the definition of  $\mathcal{A}[n]$  in (4.2), we conclude that

**Corollary 6.7.** *For all  $n \geq 1$ , the operator  $\mathcal{A}[n]$  is bounded.*

## 7. IDENTITIES SATISFIED BY THE PERIOD MATRICES

In this section, we are going to derive some identities satisfied by the period matrices.

First, we introduce the matrix  $\Pi[n]$  generalizing the matrix  $\Pi[0]$  first considered by [Nag92]. Given  $\gamma \in \text{Diff}_+(S^1)$ , we can consider the Fourier coefficients of<sup>4</sup>  $c[n]_k \gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n$  for any  $k$  and  $n$ . We define

$$\Pi[\gamma; n]_{lk} = \frac{1}{2\pi} \frac{c[n]_k}{c[n]_l} \int_0^{2\pi} \gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n e^{-i(l-n)\theta} d\theta,$$

so that

$$c[n]_k \gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n = \sum_{l \in \mathbb{Z}} \Pi[\gamma; n]_{lk} c[n]_l e^{i(l-n)\theta}.$$

The  $\mathbb{Z} \times \mathbb{Z}$  matrix  $\Pi[\gamma; n]$  is defined by  $\Pi[\gamma; n] = (\Pi[\gamma; n]_{lk})$ . It is easy to see that under group multiplication,

$$(7.1) \quad \Pi[\gamma_1; n] \Pi[\gamma_2; n] = \Pi[\gamma_2 \circ \gamma_1].$$

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<sup>4</sup>Here  $\gamma' = \frac{d\gamma}{dz}$  where  $z = e^{i\theta}$ .



Therefore

$$(7.2) \quad \Pi[\gamma^{-1}; n] = \Pi[\gamma; n]^{-1}.$$

On the other hand, since  $|\gamma(e^{i\theta})| = 1$ , we have

$$\overline{\gamma(e^{i\theta})} = \gamma(e^{i\theta})^{-1}, \quad \overline{\gamma'(e^{i\theta})} = e^{2i\theta} \gamma(e^{i\theta})^{-2} \gamma'(e^{i\theta}).$$

Therefore,

$$c[n]_k \gamma(e^{i\theta})^{-k-n} \gamma'(e^{i\theta})^n = c[n]_k \overline{\gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n} e^{-2in\theta} = \sum_{l \in \mathbb{Z}} \overline{\Pi[\gamma; n]_{lk}} c[n]_l e^{i(-l-n)\theta}.$$

This implies that

$$(7.3) \quad \Pi[\gamma; n]_{-l, -k} = \overline{\Pi[\gamma; n]_{lk}}.$$

On the other hand, since

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} (\gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n) (\gamma(e^{i\theta})^{-m+n-1} \gamma'(e^{i\theta})^{1-n}) e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \gamma(e^{i\theta})^{k-m-1} \gamma'(e^{i\theta}) e^{i\theta} d\theta \\ &= \frac{1}{2\pi i} \oint_{S^1} z^{k-m-1} dz = \delta_{km}, \end{aligned}$$

we have

$$\sum_{l \in \mathbb{Z}} \Pi[\gamma; n]_{l,k} \Pi[\gamma; 1-n]_{-l, -m} = \delta_{km}.$$

Together with (7.3), we obtain

$$\Pi[\gamma; 1-n]^* \Pi[\gamma; n] = \text{Id}.$$

In view of (7.2), we conclude that

$$\Pi[\gamma^{-1}; n] = \Pi[\gamma; 1-n]^*.$$

We re-collect these identities into the following lemma.

**Lemma 7.1.** *For any  $\gamma \in \text{Diff}_+(S^1)$  and any integer  $n$ , we have the following identities:*

- (i)  $\overline{\Pi[\gamma; n]_{lk}} = \Pi[\gamma; n]_{-l, -k}.$
- (ii)  $\Pi[\gamma^{-1}; n] = \Pi[\gamma; n]^{-1} = \Pi[\gamma; 1-n]^*.$

Now for  $n \geq 1$ , let

$$\begin{aligned} \Pi_1[\gamma; n] &= (\Pi[\gamma; n]_{l,k})_{l,k \geq n}, & \hat{\Pi}_2[\gamma; n] &= (\Pi[\gamma; n]_{-l,k})_{l \geq 1-n, k \geq n}, \\ \hat{\Pi}_3[\gamma; n] &= (\Pi[\gamma; n]_{l,-k})_{l \geq n, k \geq 1-n}, & \Pi_4[\gamma; n] &= (\Pi[\gamma; n]_{-l,-k})_{l,k \geq 1-n} \\ \Pi_2[\gamma; n] &= (\Pi[\gamma; n]_{-l,k})_{l,k \geq n}, \end{aligned}$$

$$\begin{aligned} \Pi_1[\gamma; 1-n] &= (\Pi[\gamma; 1-n]_{l,k})_{l,k \geq n}, & \hat{\Pi}_2[\gamma; 1-n] &= (\Pi[\gamma; 1-n]_{-l,k})_{l \geq 1-n, k \geq n}, \\ \hat{\Pi}_3[\gamma; 1-n] &= (\Pi[\gamma; 1-n]_{l,-k})_{l \geq n, k \geq 1-n}, & \Pi_4[\gamma; 1-n] &= (\Pi[\gamma; 1-n]_{-l,-k})_{l,k \geq 1-n} \\ \Pi_2[\gamma; 1-n] &= (\Pi[\gamma; 1-n]_{-l,k})_{l,k \geq n}, \end{aligned}$$

By Lemma 7.1, it follows immediately that

**Lemma 7.2.** *For any  $\gamma \in \text{Diff}_+(S^1)$  and any integer  $n$ , we have the following identities:*

- (i)  $\Pi_1[\gamma; n]^* = \Pi_1[\gamma^{-1}; 1 - n]$ ,
- (ii)  $\hat{\Pi}_2[\gamma; n]^* = \hat{\Pi}_3[\gamma^{-1}; 1 - n]$ ,
- (iii)  $\Pi_2[\gamma; n]^* = \overline{\Pi_2[\gamma^{-1}; 1 - n]}$ .

**Proposition 7.3.** *For every point  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ , we have the following relations.*

**I.** *For any integer  $n$ ,*

$$\begin{aligned} A[\gamma; n] &= \Pi_1[\gamma^{-1}; n]^{-1}, & D[\gamma; n] &= \overline{\Pi_1[\gamma; n]}^{-1} \\ B[\gamma; n] &= \Pi_2[\gamma^{-1}; n] \Pi_1[\gamma^{-1}; n]^{-1}, & C[\gamma; n] &= \overline{\Pi_2[\gamma; n]} \overline{\Pi_1[\gamma; n]}^{-1} \\ \hat{B}[\gamma; n] &= \hat{\Pi}_2[\gamma^{-1}; n] \Pi_1[\gamma^{-1}; n]^{-1}, & \hat{C}[\gamma; n] &= \overline{\hat{\Pi}_2[\gamma; n]} \overline{\Pi_1[\gamma; n]}^{-1}. \end{aligned}$$

**II.** *For any integer  $n \geq 1$ ,*

$$(7.4) \quad \mathbb{A}[1 - n] = \overline{\Pi_4[\gamma^{-1}; 1 - n]}^{-1}.$$

*Proof.* For  $n \geq 1$ , since  $f \circ \gamma^{-1} = g$ , we find that restricted to  $S^1$ ,

$$(U[n]_k \circ f(f')^n) \circ \gamma^{-1} ((\gamma^{-1})')^n = U[n]_k \circ g(g')^n.$$

Using the expansion of  $U[n]_k \circ f(f')^n$  and  $U[n]_k \circ g(g')^n$  give

$$(7.5) \quad \Pi_1[\gamma^{-1}; n] A[\gamma; n] = \text{Id}, \quad \Pi_2[\gamma^{-1}; n] A[\gamma; n] = B[\gamma; n].$$

The other identities are proved similarly.  $\square$

From this, Proposition 6.4, Corollary 6.5 and the fact that  $\mathfrak{P}[n]$  and  $\mathfrak{M}[n]$  are invertible, we conclude that

**Lemma 7.4.** *For any integer  $n \geq 1$ ,  $A[1 - n]$ ,  $A[n]$ ,  $\mathfrak{A}[n]$ ,  $D[1 - n]$ ,  $D[n]$ ,  $\mathfrak{D}[n]$ ,  $\Pi_1[1 - n]$ ,  $\Pi_1[n]$  define invertible operators on  $\ell^2$ .*

From part **II** of Proposition 7.3, we also have

**Corollary 7.5.** *For any integer  $n \geq 1$ ,  $\mathcal{D}[n]$  is a bounded operator.*

*Proof.* Let

$$\begin{aligned} \mathbb{A}_1[1 - n] &= (\mathbb{A}[1 - n]_{l,k})_{l,k \geq n}, & \mathbb{A}_2[1 - n] &= (\mathbb{A}[1 - n]_{lk})_{1-n \leq l \leq n-1, k \geq n}, \\ \mathbb{A}_3[1 - n] &= (\mathbb{A}[1 - n]_{l,k})_{l \geq n, 1-n \leq k \leq n-1}, & \mathbb{A}_4[1 - n] &= (\mathbb{A}[1 - n]_{lk})_{1-n \leq l, k \leq n-1}, \\ \Pi_5[\gamma; 1 - n] &= (\Pi[\gamma; 1 - n]_{lk})_{1-n \leq l \leq n-1, k \geq n}, & \Pi_6[\gamma; 1 - n] &= (\Pi[\gamma; 1 - n]_{lk})_{l \geq n, 1-n \leq k \leq n-1}, \\ \Pi_7[\gamma; 1 - n] &= (\Pi[\gamma; 1 - n]_{lk})_{1-n \leq l, k \leq n-1}. \end{aligned}$$

$\mathbb{A}_1[1 - n]$  is the kernel of  $\mathcal{D}[n]$  with respect to standard bases. (7.4) implies that

$$\begin{pmatrix} \mathbb{A}_1[1 - n] & \mathbb{A}_3[1 - n] \\ \mathbb{A}_2[1 - n] & \mathbb{A}_4[1 - n] \end{pmatrix} \begin{pmatrix} \Pi_1[\gamma^{-1}; 1 - n] & \Pi_6[\gamma^{-1}; 1 - n] \\ \Pi_5[\gamma^{-1}; 1 - n] & \Pi_7[\gamma^{-1}; 1 - n] \end{pmatrix} = \text{Id}.$$

Therefore,

$$\mathbb{A}_1[1 - n] \Pi_1[\gamma^{-1}; 1 - n] + \mathbb{A}_3[1 - n] \Pi_5[\gamma^{-1}; 1 - n] = \text{Id}.$$

Since  $\mathbb{A}_3[1 - n] \Pi_5[\gamma^{-1}; 1 - n]$  defines a finite rank operator, it is bounded. Therefore, the boundedness of the operator defined by  $\Pi_1[\gamma^{-1}; 1 - n]^{-1}$  implies the boundedness of  $\mathcal{D}[n]$ .  $\square$

To derive Grunsky-type identities, for  $n \geq 1$  we differentiate both sides of the formula

$$c[1-n]_k \gamma(e^{i\theta})^{k+n-1} \gamma'(e^{i\theta})^{1-n} = \sum_{l \in \mathbb{Z}} \Pi[\gamma; 1-n]_{lk} c[1-n]_l e^{i(l+n-1)\theta}$$

with respect to  $z = e^{i\theta}$   $(2n-1)$  times. Lemma 6.1 gives us

$$\begin{aligned} & (\text{sgn}_n k) c[n]_k \gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n \\ &= \sum_{|l| \geq n} \Pi[\gamma; 1-n]_{lk} (\text{sgn}_n l) c[n]_l e^{i(l-n)\theta} \\ & \quad + \xi[\gamma; n]_2 (e^{i\theta}) \sum_{l \in \mathbb{Z}} \Pi[\gamma; 1-n]_{lk} c[1-n]_l (l+n-1) \dots (l-n+3) e^{i(l-n+2)\theta} \\ & \quad + \xi[\gamma; n]_3 (e^{i\theta}) \sum_{l \in \mathbb{Z}} \Pi[\gamma; 1-n]_{lk} c[1-n]_l (l+n-1) \dots (l-n+4) e^{i(l-n+3)\theta} \\ & \quad + \dots + \xi[\gamma; n]_{2n-1} (e^{i\theta}) \sum_{l \in \mathbb{Z}} \Pi[\gamma; 1-n]_{lk} c[1-n]_l e^{i(l+n-1)\theta}. \end{aligned}$$

Therefore, for all  $|k| \geq n$ ,

$$\begin{aligned} (7.6) \quad & (\text{sgn}_n k) \sum_{l \in \mathbb{Z}} \Pi[\gamma; n]_{lk} c[n]_l e^{i(l-n)\theta} \\ &= \sum_{|l| \geq n} \Pi[\gamma; 1-n]_{lk} (\text{sgn}_n l) c[n]_l e^{i(l-n)\theta} + \sum_{l \in \mathbb{Z}} \mathfrak{S}[\gamma; n]_{lj} \Pi[\gamma; 1-n]_{jk} (\text{sgn}_n j) c[n]_l e^{i(l-n)\theta}, \end{aligned}$$

and for  $|k| < n$ ,

$$(7.7) \quad 0 = \sum_{|l| \geq n} \Pi[\gamma; 1-n]_{lk} (\text{sgn}_n l) c[n]_l e^{i(l-n)\theta} + \sum_{l \in \mathbb{Z}} \mathfrak{S}[n]_{lj} \Pi[\gamma; 1-n]_{jk} (\text{sgn}_n j) c[n]_l e^{i(l-n)\theta},$$

where

$$\mathfrak{S}[\gamma; n]_{lj} = (\text{sgn}_n j) \sum_{m=2}^{2n-1} \Xi[\gamma; n; m]_{l-j} \frac{c[2]_{l-j} c[1-n]_j}{c[n]_l} (j+n-1) \dots (j-n+m+1),$$

and

$$\xi[\gamma; n]_m (e^{i\theta}) = \sum_{k \in \mathbb{Z}} \Xi[\gamma; n; m]_k c[2]_k e^{i(k-m)\theta}.$$

*Remark 7.6.* When  $n = 2$ ,  $\xi[\gamma; 2]_2 = 2\mathcal{S}(\gamma)$ ,  $\xi[\gamma; 2]_3 = \mathcal{S}(\gamma)'$ . Let

$$\mathcal{S}(\gamma)(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \gamma_k c[2]_k e^{i(k-2)\theta}.$$

$|\gamma(e^{i\theta})| = 1$  implies that

$$\gamma_{-k} = \bar{\gamma}_k.$$

A straightforward computation gives

$$\mathfrak{S}[\gamma; 2]_{lj} = (\text{sgn}_2 j) \frac{c[2]_{l-j} c[-1]_j}{c[2]_l} (j+l) \gamma_{l-j}.$$

For any integer  $n$ , let

$$\tilde{\Pi}[\gamma; n] = \left( \tilde{\Pi}[\gamma; n]_{lk} \right)_{l, k \in \mathbb{Z}},$$

where

$$\tilde{\Pi}[\gamma; n]_{lk} = \begin{cases} \Pi[\gamma; n]_{lk}, & \text{if } |k| \geq n, \\ 0, & \text{if } |k| < n. \end{cases}$$

Also for  $n \geq 1$ , define

$$J = (J_{lk}), \quad \text{where} \quad J_{lk} = \delta_{lk} \operatorname{sgn}_n k,$$

and

$$\mathfrak{T}[\gamma; n] = I[n] + \mathfrak{S}[\gamma; n] = (I[n]_{lk} + \mathfrak{S}[\gamma; n]_{lk})_{l, k \in \mathbb{Z}},$$

where

$$I[n]_{lk} = \begin{cases} 1, & \text{if } k = l \geq n \text{ or } k = l \leq -n, \\ 0, & \text{otherwise.} \end{cases}$$

Equations (7.6) and (7.7) say that

$$(7.8) \quad \tilde{\Pi}[\gamma; n] = \mathfrak{T}[\gamma; n] J \Pi[\gamma; 1-n] J = \mathfrak{T}[\gamma; n] J \tilde{\Pi}[\gamma; 1-n] J.$$

Multiplying  $J \Pi[\gamma^{-1}; 1-n] J = J \Pi[\gamma; n]^* J$  on the right of both sides, we find that

$$\tilde{\Pi}[\gamma; n] J \Pi[\gamma; n]^* J = \mathfrak{T}[\gamma; n].$$

Since the  $|k| < n$  columns in  $\tilde{\Pi}[\gamma; n]$  are identically zeros, the  $|k| < n$  rows of  $J \Pi[\gamma; n]^* J$  do not contribute anything to the product  $\tilde{\Pi}[\gamma; n] J \Pi[\gamma; n]^* J$ . Therefore, we can replace the  $|k| < n$  rows in  $J \Pi[\gamma; n]^* J$  by zeros and  $\tilde{\Pi}[\gamma; n] J \Pi[\gamma; n]^* J = \tilde{\Pi}[\gamma; n] J \tilde{\Pi}[\gamma; n]^* J$ . This implies that

$$(7.9) \quad \mathfrak{T}[\gamma; n] = \tilde{\Pi}[\gamma; n] J \tilde{\Pi}[\gamma; n]^* J,$$

and therefore

$$(7.10) \quad \mathfrak{T}[\gamma; n]^* = J \mathfrak{T}[\gamma; n] J.$$

Let

$$\begin{aligned} \mathfrak{S}_1[\gamma; n] &= (\mathfrak{S}[\gamma; n]_{lk})_{l, k \geq n}, & \mathfrak{S}_2[\gamma; n] &= (\mathfrak{S}[\gamma; n]_{-l, k})_{l, k \geq n}, \\ \mathfrak{S}_3[\gamma; n] &= (\mathfrak{S}[\gamma; n]_{l, -k})_{l, k \geq n}, & \mathfrak{S}_4[\gamma; n] &= (\mathfrak{S}[\gamma; n]_{-l, -k})_{l, k \geq n}. \end{aligned}$$

Equation (7.10) implies that

$$\mathfrak{S}_1[n]^* = \mathfrak{S}_1[n], \quad \mathfrak{S}_2[n]^* = -\mathfrak{S}_3[n], \quad \mathfrak{S}_4[n]^* = \mathfrak{S}_4[n].$$

Now, by deleting the  $|l| \leq n-1$  rows and  $|k| \leq n-1$  columns of the matrices  $\mathfrak{T}[\gamma; n]$ , we find from (7.9) and (7.3) that

**Lemma 7.7.** *For every integer  $n \geq 1$ , we have the following identity.*

$$(7.11) \quad \begin{pmatrix} \operatorname{Id} & 0 \\ 0 & \operatorname{Id} \end{pmatrix} + \begin{pmatrix} \mathfrak{S}_1[n] & \mathfrak{S}_3[n] \\ \mathfrak{S}_2[n] & \mathfrak{S}_4[n] \end{pmatrix} = \begin{pmatrix} \Pi_1[n] & \overline{\Pi_2[n]} \\ \Pi_2[n] & \overline{\Pi_1[n]} \end{pmatrix} \begin{pmatrix} \Pi_1[n]^* & -\Pi_2[n]^* \\ -\Pi_2[n]^* & \Pi_1[n]^* \end{pmatrix}.$$

Comparing both sides, we have

$$(7.12) \quad \begin{aligned} \text{Id} + \mathfrak{S}_1[n] &= \Pi_1[n] \Pi_1[n]^* - \overline{\Pi_2[n]} \overline{\Pi_2[n]}^* \\ \mathfrak{S}_3[n] &= \overline{\Pi_2[n]} \overline{\Pi_1[n]}^* - \Pi_1[n] \Pi_2[n]^* \end{aligned}$$

and

$$(7.13) \quad \mathfrak{S}_4[n] = \overline{\mathfrak{S}_1[n]}, \quad \mathfrak{S}_3[n] = \overline{\mathfrak{S}_2[n]}.$$

*Remark 7.8.* When  $n = 1$ , we have  $\mathfrak{S}[1] = 0$ . Therefore, (7.11) says that

$$(7.14) \quad \begin{pmatrix} \Pi_1[1] & \overline{\Pi_2[1]} \\ \Pi_2[1] & \overline{\Pi_1[1]} \end{pmatrix} \begin{pmatrix} \Pi_1[1]^* & -\Pi_2[1]^* \\ -\overline{\Pi_2[1]}^* & \overline{\Pi_1[1]}^* \end{pmatrix} = \text{Id}.$$

On the other hand, by removing the 0-th row of (7.8), we find that

$$(7.15) \quad \begin{pmatrix} \Pi_1[\gamma; 1] & \overline{\Pi_2[\gamma; 1]} \\ \Pi_2[\gamma; 1] & \overline{\Pi_1[\gamma; 1]} \end{pmatrix} = \begin{pmatrix} \Pi_1[\gamma; 0] & -\overline{\Pi_2[\gamma; 0]} \\ -\Pi_2[\gamma; 0] & \overline{\Pi_1[\gamma; 0]} \end{pmatrix}.$$

Together with (7.14), this gives

$$\begin{pmatrix} \Pi_1[\gamma; 0] & \overline{\Pi_2[\gamma; 0]} \\ \Pi_2[\gamma; 0] & \overline{\Pi_1[\gamma; 0]} \end{pmatrix} \begin{pmatrix} \Pi_1[\gamma; 0]^* & -\Pi_2[\gamma; 0]^* \\ -\overline{\Pi_2[\gamma; 0]}^* & \overline{\Pi_1[\gamma; 0]}^* \end{pmatrix} = \text{Id},$$

a well known identity (see e.g. [Nag92]). From this equation and Proposition 7.3, we can derive the Grunsky equality for the pair  $(f, g)$ .

Now we derive the Grunsky-type identities for  $(f, g)$  from (7.12). For  $n \geq 1$ , equation (7.12) gives

$$\begin{aligned} \Pi_1[\gamma^{-1}; n]^{-1} (\Pi_1[\gamma^{-1}; n]^{-1})^* &= \text{Id} - \Pi_1[\gamma^{-1}; n]^{-1} \mathfrak{S}_1[\gamma^{-1}; n] (\Pi_1[\gamma^{-1}; n]^{-1})^* \\ &\quad - \Pi_1[\gamma^{-1}; n]^{-1} \overline{\Pi_2[\gamma^{-1}; n]} \overline{\Pi_2[\gamma^{-1}; n]}^* (\Pi_1[\gamma^{-1}; n]^{-1})^*, \end{aligned}$$

$$\begin{aligned} &\Pi_1[\gamma^{-1}; n]^{-1} \mathfrak{S}_3[\gamma^{-1}; n] \left( \overline{\Pi_1[\gamma^{-1}; n]^{-1}} \right)^* \\ &= \Pi_1[\gamma^{-1}; n] \overline{\Pi_2[\gamma^{-1}; n]} - \Pi_2[\gamma^{-1}; n]^* \left( \overline{\Pi_1[\gamma^{-1}; n]^{-1}} \right)^*. \end{aligned}$$

By Lemma 7.2, Proposition 7.3, (3.7) and (3.11), we have

$$\begin{aligned} \Pi_1[\gamma^{-1}; n]^{-1} &= A[\gamma; n] = D[\gamma; 1 - n]^T = A[\gamma^{-1}; 1 - n]^* = \overline{D[\gamma^{-1}; n]}, \\ \overline{\Pi_2[\gamma^{-1}; n]}^* (\Pi_1[\gamma^{-1}; n]^{-1})^* &= \Pi_2[\gamma; 1 - n] \Pi_1[\gamma; 1 - n]^{-1} = \overline{C[\gamma; 1 - n]} = B[\gamma^{-1}; 1 - n]. \end{aligned}$$

Therefore, we obtain

**Proposition 7.9.** *For any integer  $n \geq 1$  and  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ , we have the following identities.*

$$(7.16) \quad \begin{aligned} A[\gamma; n] A[\gamma; n]^* &= \text{Id} - A[\gamma; n] \mathfrak{S}_1[\gamma^{-1}; n] A[\gamma; n]^* - C[\gamma; 1 - n]^T \overline{C[\gamma; 1 - n]}, \\ A[\gamma; 1 - n]^* A[\gamma; 1 - n] &= \text{Id} - A[\gamma; 1 - n]^* \mathfrak{S}_1[\gamma; n] A[\gamma; 1 - n] - B[\gamma; 1 - n]^* B[\gamma; 1 - n], \\ D[\gamma; n] D[\gamma; n]^* &= \text{Id} - D[\gamma; n] \mathfrak{S}_4[\gamma; n] D[\gamma; n]^* - B[\gamma; 1 - n]^T \overline{B[\gamma; 1 - n]}, \\ D[\gamma; 1 - n]^* D[\gamma; 1 - n] &= \text{Id} - D[\gamma; 1 - n]^* \mathfrak{S}_4[\gamma^{-1}; n] D[\gamma; 1 - n] - C[\gamma; 1 - n]^* C[\gamma; 1 - n], \\ A[\gamma; n] \mathfrak{S}_3[\gamma^{-1}; n] A[\gamma; n]^T &= C[\gamma; 1 - n]^T - C[\gamma; 1 - n], \\ D[\gamma; n]^T \mathfrak{S}_2[\gamma; n] D[\gamma; n] &= B[\gamma; 1 - n]^T - B[\gamma; 1 - n]. \end{aligned}$$

8. THE FUNCTIONS  $\mathfrak{F}_n$  AND  $\mathfrak{G}_n$  ON  $S^1 \setminus \text{Diff}_+(S^1)$ .

In this section, we want to show that we can define real-valued functions  $\mathfrak{F}_n : S^1 \setminus \text{Diff}_+(S^1) \rightarrow \mathbb{R}$  and  $\mathfrak{G}_n : S^1 \setminus \text{Diff}_+(S^1) \rightarrow \mathbb{R}$  by  $\log \det A[\gamma; n]A[\gamma; n]^*$  and  $\log \det \mathfrak{A}[n]\mathfrak{A}[n]^*$  respectively.

First, we have the following propositions:

**Proposition 8.1.** *For any  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ , the matrix  $\mathfrak{S}_1[\gamma; n]$  defines a trace class operator on  $\ell^2$ .*

*Proof.* Since  $\mathfrak{S}_1[n]^* = \mathfrak{S}_1[n]$ , we can write

$$\mathfrak{S}_1[n] = D[n] + \mathfrak{L}[n] + \mathfrak{L}[n]^*,$$

where  $D[n]$  is a real-valued diagonal matrix and  $\mathfrak{L}[n]$  is a strictly lower triangular matrix. From the previous section, we find that

$$\begin{aligned} D[n] &= \sum_{m=2}^{2n-1} D_m[n] = \sum_{m=2}^{2n-1} (D[n]_{lk})_{l,k \geq n}, \\ \mathfrak{L}[n] &= \sum_{m=2}^{2n-1} \mathfrak{L}_m[n] = \sum_{m=2}^{2n-1} (\mathfrak{L}_m[n]_{lk})_{l,k \geq n}, \end{aligned}$$

where

$$\begin{aligned} D_m[n]_{lk} &= \delta_{l,k} \Xi[\gamma; n; m]_0 \frac{c[2]_0 c[1-n]_k}{c[n]_k} \frac{(k+n-1)!}{(k-n+m)!}, \\ \mathfrak{L}_m[n]_{lk} &= \begin{cases} \Xi[\gamma; n; m]_{l-k} \frac{c[2]_{l-k} c[1-n]_k}{c[n]_l} \frac{(k+n-1)!}{(k-n+m)!}, & \text{if } l > k \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

and

$$\xi[\gamma; n]_m(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \Xi[\gamma; n; m]_k c[2]_k e^{i(k-m)\theta}.$$

It suffices to show that each of the matrices  $D_m[n]$ ,  $\mathfrak{L}_m[n]$ ,  $2 \leq m \leq 2n-1$  defines a trace class operator on  $\ell^2$ . By (3.2),

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{c[1-n]_k}{c[n]_k} \frac{(k+n-1)!}{(k-n+m+1)!} &= \sum_{k=n}^{\infty} \frac{1}{(k-n+1)(k-n+2) \dots (k-n+m)} \\ &\leq \sum_{k=n}^{\infty} \frac{1}{(k-n+1)(k-n+2)} < \infty, \end{aligned}$$

therefore  $D_m[n]$  defines a trace class operator for all  $2 \leq m \leq 2n-1$ .

To show that  $\mathfrak{L}_m[n]$  defines a trace class operator, we define

$$\Lambda = (\Lambda_{lk})_{l,k \geq n}, \quad \text{where} \quad \Lambda_{l,k} = \delta_{l,k+1}$$

and write

$$\begin{aligned} (8.1) \quad \mathfrak{L}_m[n] &= c[2]_1 \xi[\gamma; n; m]_1 \Lambda D_{1,m}[n] + c[2]_2 \xi[\gamma; n; m]_2 \Lambda^2 D_{2,m}[n] \\ &\quad + \dots + c[2]_{m-1} \xi[\gamma; n; m]_{m-1} \Lambda^{m-1} D_{m-1,m}[n] + \hat{\mathfrak{L}}_m[n], \end{aligned}$$

where

$$D_{j,m}[n] = \left( \delta_{l,k} \frac{c[1-n]_k}{c[n]_{k+j}} \frac{(k+n-1)!}{(k-n+m)!} \right)_{l,k \geq n},$$

$$\hat{\mathfrak{L}}_m[n] = (\hat{\mathfrak{L}}_m[n]_{l,k}), \quad \hat{\mathfrak{L}}_m[n]_{l,k} = \begin{cases} \mathfrak{L}_m[n]_{l,k}, & \text{if } l \geq k + m \\ 0, & \text{otherwise} \end{cases}.$$

Since  $c[n]_{k+j} \geq c[n]_k$  for all  $j \geq 1$ , we can show as the case of  $D_m[n]$  that the matrix  $D_{j,m}$  is of trace class for all  $1 \leq j < m \leq 2n - 1$ . On the other hand, since the function  $\xi[\gamma; n]_m$  is smooth on  $S^1$ , we can show as in Proposition 6.4 that  $\hat{\mathfrak{L}}_m[n]$  defines a trace class operator. Since  $\Lambda$  defines a bounded operator, (8.1) shows that  $\mathfrak{L}_m[n]$  defines a trace class operator. This completes the proof.  $\square$

**Proposition 8.2.** *For any  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$  and any integer  $n \geq 1$ , the matrix  $C[1 - n]$  defines a Hilbert Schmidt operator on  $\ell^2$ .*

*Proof.* For  $n = 1$ , the result is already known (see e.g. [TT06]). Hence we assume that  $n \geq 2$ .

From the definition (4.2), to show that  $C[1 - n]$  defines a Hilbert-Schmidt operator is equivalent to showing that the operator  $\mathcal{C}[n]$  is Hilbert-Schmidt. By Lemma 6.3, we need to prove that

$$\iint_{\mathbb{D}} \iint_{\mathbb{D}} |\mathcal{C}[n](z, w)|^2 \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w < \infty.$$

For this purpose, we use variational techniques. From (4.3), we have

$$(8.2) \quad \mathcal{C}[n](z, w) = -\alpha_n \frac{d^{2n-1}}{dw^{2n-1}} \left( \frac{f'(z)^n f'(w)^{1-n}}{f(z) - f(w)} - \frac{1}{z - w} \right).$$

Given a point  $\gamma \in S^1 \setminus \text{Diff}_+(S^1)$ , we can joint it to the origin  $\text{id} \in S^1 \setminus \text{Diff}_+(S^1)$  by a smooth curve  $\gamma_t$ ,  $t \in [0, 1]$  such that  $\gamma_0 = \text{id}$  and  $\gamma_1 = \gamma$ . We have the following well-known fact [Kir87]:

**Lemma 8.3.** *Given a smooth curve  $\gamma_t \in S^1 \setminus \text{Diff}_+(S^1)$  and its associated pair  $(f^t, g_t)$ , we have*

$$\left. \frac{df^{t+s}}{ds} \right|_{s=0} (z) = \frac{1}{2\pi i} \oint_{S^1} \frac{f^t(z)^2 g_t'(\zeta)^2 u_t(\zeta)}{g_t(\zeta)^2 (g_t(\zeta) - f^t(z))} d\zeta,$$

where

$$u_t(\zeta) = \left. \frac{d\gamma_{t+s}}{ds} \right|_{s=0} \circ \gamma_t^{-1}(\zeta)$$

is a smooth function on  $S^1$ .

For  $\zeta \in S^1$ , let

$$u_t(\zeta) = \sum_{k \in \mathbb{Z}} c_k(t) \zeta^{k+1}$$

and define

$$v_t(\zeta) = \sum_{k=1}^{\infty} c_k(t) \zeta^{k+1}, \quad \zeta \in S^1.$$

It is easy to see that

$$\frac{1}{2\pi i} \oint_{S^1} \frac{f^t(z)^2 g_t'(\zeta)^2 u_t(\zeta)}{g_t(\zeta)^2 (g_t(\zeta) - f^t(z))} d\zeta = \frac{1}{2\pi i} \oint_{S^1} \frac{f^t(z)^2 g_t'(\zeta)^2 v_t(\zeta)}{g_t(\zeta)^2 (g_t(\zeta) - f^t(z))} d\zeta.$$

Now a straight-forward computation gives

**Lemma 8.4.**

$$(8.3) \quad \left. \frac{d}{ds} \right|_{s=0} \frac{(f^{t+s})'(z)^n (f^{t+s})'(w)^{1-n}}{f^{t+s}(z) - f^{t+s}(w)} = \frac{n}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) g'_t(\zeta)^2 (f^t)'(z)^n (f^t)'(w)^{1-n}}{(g_t(\zeta) - f^t(w))(g_t(\zeta) - f^t(z))^2} d\zeta \\ - \frac{1-n}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) g'_t(\zeta)^2 (f^t)'(z)^n (f^t)'(w)^{1-n}}{(g_t(\zeta) - f^t(w))^2 (g_t(\zeta) - f^t(z))} d\zeta.$$

Let

$$\frac{g'_t(\zeta)^n (f^t)'(w)^{1-n}}{g_t(\zeta) - f^t(w)} = \frac{1}{\alpha_n} \sum_{k=1-n}^{\infty} \sum_{l=1-n}^{\infty} \mathbb{A}_t[1-n]_{lk} c[n]_k \zeta^{-k-n} c[1-n]_l w^{l+n-1}, \\ \frac{g'_t(\zeta)^{n+1} (f^t)'(w)^{1-n}}{(g_t(\zeta) - f^t(w))^2} = \frac{1}{\alpha_n} \sum_{k=1-n}^{\infty} \sum_{l=1-n}^{\infty} \mathbb{H}_t[n]_{lk} c[n+1]_k \zeta^{-k-n-1} c[1-n]_l w^{l+n-1},$$

and define

$$(8.4) \quad \hat{\mathcal{D}}_t[n](\zeta, w) = \frac{1}{\alpha_n} \sum_{k=1-n}^{\infty} \sum_{l=n}^{\infty} \mathbb{A}_t[1-n]_{lk} c[n]_k \zeta^{-k-n} c[n]_l w^{l-n}, \\ H_t[n](\zeta, w) = \frac{1}{\alpha_n} \sum_{k=1-n}^{\infty} \sum_{l=n}^{\infty} \mathbb{H}_t[n]_{lk} c[n+1]_k \zeta^{-k-n-1} c[n]_l w^{l-n}.$$

Comparing (8.3) with (8.2), we find that

**Corollary 8.5.**

$$\left. \frac{d}{ds} \right|_{s=0} \mathcal{C}[n]_{t+s}(z, w) = \alpha_n \left( -\frac{n}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) g'_t(\zeta)^{2-n} (f^t)'(z)^n}{(g_t(\zeta) - f^t(z))^2} \hat{\mathcal{D}}[n]_t(\zeta, w) d\zeta \right. \\ \left. + \frac{1-n}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) g'_t(\zeta)^{1-n} (f^t)'(z)^n}{g_t(\zeta) - f^t(z)} H_t(\zeta, w) d\zeta \right).$$

Now we prove the following.

**Lemma 8.6.**

$$\iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \left. \frac{d}{ds} \right|_{s=0} \mathcal{C}[n]_{t+s}(z, w) \right|^2 \rho(z)^{1-n} d^2 z \rho(w)^{1-n} d^2 w \leq \mathfrak{K}_t \sum_{k=1}^{\infty} k^{2n+2} |c_k(t)|^2$$

for some constant  $\mathfrak{K}_t$  that depends continuously on  $t$ .

*Proof.* By (4.4), we can write

$$\hat{\mathcal{D}}_t(\zeta, w) = R_t[n](\zeta, w) + \mathcal{D}_t[n](\zeta, w),$$

where

$$R_t[n](\zeta, w) = \sum_{k=1-n}^{n-1} \mathfrak{U}_t[n]_k(w) c[n]_k \zeta^{-k-n}.$$

Since

$$\frac{g'_t(\zeta)^{n+1} (f^t)'(w)^{1-n}}{(g_t(\zeta) - f^t(w))^2} = -\frac{\partial}{\partial \zeta} \left( \frac{g'_t(\zeta)^n (f^t)'(w)^{1-n}}{g_t(\zeta) - f^t(w)} \right) + \frac{g''_t(\zeta)}{g'_t(\zeta)} \frac{g'_t(\zeta)^n (f^t)'(w)^{1-n}}{g_t(\zeta) - f^t(w)},$$



we have from the definition (8.4),

$$\begin{aligned} H_t(\zeta, w) &= -\frac{\partial \hat{\mathcal{D}}_t[n](\zeta, w)}{\partial \zeta} + \frac{g_t''(\zeta)}{g_t'(\zeta)} \hat{\mathcal{D}}_t[n]_t(\zeta, w) \\ &= -\frac{\partial R_t[n](\zeta, w)}{\partial \zeta} + \frac{g_t''(\zeta)}{g_t'(\zeta)} R_t[n](\zeta, w) - \frac{\partial \mathcal{D}_t[n](\zeta, w)}{\partial \zeta} + \frac{g_t''(\zeta)}{g_t'(\zeta)} \mathcal{D}_t[n](\zeta, w). \end{aligned}$$

By standard reproducing formulas, we have

$$\begin{aligned} \mathcal{D}_t[n](\zeta, w) &= \beta_n \iint_{\mathbb{D}^*} \frac{\mathcal{D}_t[n](v, w) \rho(v)^{1-n}}{(1 - \zeta \bar{v})^{2n}} d^2 v, \\ \frac{g_t''(\zeta)}{g_t'(\zeta)} &= \frac{1}{\pi} \iint_{\mathbb{D}^*} \frac{g_t''(\eta)}{g_t'(\eta)} \frac{1}{(1 - \bar{\eta} \zeta)^2} d^2 \eta. \end{aligned}$$

On the other hand, by (3.10) and the definition of  $\mathcal{A}[n]$  in (4.2), we can verify directly that

$$\begin{aligned} \frac{g_t'(\zeta)^{1-n} (f^t)'(z)^n}{g_t(\zeta) - f^t(z)} &= \frac{1}{\alpha_n} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} A_t[n]_{lk} c[1-n]_k \zeta^{-k+n-1} c[n]_l z^{l-n} \\ &= \iint_{\mathbb{D}^*} \frac{\mathcal{A}_t[n](u, z) \rho(u)^{1-n}}{\bar{u}^{2n-1} (\bar{u} \zeta - 1)} d^2 u. \end{aligned}$$

Differentiating with respect to  $\zeta$ , we have

$$\frac{g_t'(\zeta)^{2-n} f_t'(z)^n}{(g_t(\zeta) - f^t(z))^2} = \iint_{\mathbb{D}} \frac{\mathcal{A}_t[n](u, z) \rho(u)^{1-n}}{(\bar{u} \zeta - 1)^2 \bar{u}^{2n-2}} d^2 u + \frac{g_t''(\zeta)}{g_t'(\zeta)} \iint_{\mathbb{D}} \frac{\mathcal{A}[n]_t(u, z) \rho(u)^{1-n}}{(\bar{u} \zeta - 1) \bar{u}^{2n-1}} d^2 u.$$

From these, we can write

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) g_t'(\zeta)^{2-n} (f^t)'(z)^n}{(g_t(\zeta) - f^t(z))^2} \hat{\mathcal{D}}_t(\zeta, w) d\zeta \\ &= \frac{1}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) R_t[n](\zeta, w)}{(\bar{u} \zeta - 1)^2 \bar{u}^{2n-2}} \rho(u)^{1-n} d^2 u d\zeta \\ &\quad + \frac{1}{2\pi^2 i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) R_t[n](\zeta, w)}{\bar{u}^{2n-1} (\bar{u} \zeta - 1) (1 - \bar{\eta} \zeta)^2} \frac{g_t''(\eta)}{g_t'(\eta)} \rho(u)^{1-n} d^2 \eta d^2 u d\zeta \\ &\quad + \frac{\beta_n}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) \mathcal{D}_t[n](v, w)}{\bar{u}^{2n-2} (\bar{u} \zeta - 1)^2 (1 - \bar{v} \zeta)^{2n}} \rho(v)^{1-n} \rho(u)^{1-n} d^2 v d^2 u d\zeta \\ &\quad + \frac{\beta_n}{2\pi^2 i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) \mathcal{D}_t[n](v, w)}{\bar{u}^{2n-1} (\bar{u} \zeta - 1) (1 - \bar{v} \zeta)^{2n} (1 - \bar{\eta} \zeta)^2} \frac{g_t''(\eta)}{g_t'(\eta)} \\ &\quad \rho(u)^{1-n} \rho(v)^{1-n} d^2 \eta d^2 v d^2 u d\zeta, \end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) g'_t(\zeta)^{1-n} (f^t)'(z)^n}{g_t(\zeta) - f^t(z)} H_t(\zeta, w) d\zeta \\
&= -\frac{1}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z)}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)} \frac{\partial R_t[n](\zeta, w)}{\partial \zeta} \rho(u)^{1-n} d^2 u d^2 \zeta \\
&+ \frac{1}{2\pi^2 i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z)}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \zeta\bar{\eta})^2} \frac{g''_t(\eta)}{g'_t(\eta)} R_t[n](\zeta, w) \rho(u)^{1-n} d^2 \eta d^2 u d\zeta \\
&- \frac{2n\beta_n}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) \mathcal{D}_t[n](v, w) \bar{v}}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{v}\zeta)^{2n+1}} \rho(u)^{1-n} \rho(v)^{1-n} d^2 u d^2 v d\zeta \\
&+ \frac{\beta_n}{2\pi^2 i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) \mathcal{D}_t[n](v, w)}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{v}\zeta)^{2n}(1 - \zeta\bar{\eta})^2} \frac{g''_t(\eta)}{g'_t(\eta)} \\
&\quad \rho(u)^{1-n} \rho(v)^{1-n} d^2 \eta d^2 u d^2 v d\zeta.
\end{aligned}$$

Since from Corollary 6.7 and Corollary 7.5, both  $\mathcal{A}_t[n]$  and  $\mathcal{D}_t[n]$  are bounded operators, we have

$$\begin{aligned}
& \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{\beta_n}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) \mathcal{D}_t[n](v, w)}{\bar{u}^{2n-2}(\bar{u}\zeta - 1)^2(1 - \bar{v}\zeta)^{2n}} \rho(v)^{1-n} \rho(u)^{1-n} d^2 v d^2 u d\zeta \right|^2 \\
&\quad \times \rho(z)^{1-n} d^2 z \rho(w)^{1-n} d^2 w \\
&\leq \kappa_{1,t} \kappa_{2,t} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) \beta_n}{\bar{u}^{2n-2}(\bar{u}\zeta - 1)^2(1 - \bar{v}\zeta)^{2n}} d\zeta \right|^2 \rho(v)^{1-n} \rho(u)^{1-n} d^2 v d^2 u \\
&= \kappa_{1,t} \kappa_{2,t} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \left| \sum_{k=n}^{\infty} \sum_{m=n}^{\infty} c_{k+m}(t) c[n]_m^2 (k - n + 1) \bar{v}^{-m-n} \bar{u}^{-k-n} \right|^2 \rho(v)^{1-n} \rho(u)^{1-n} d^2 v d^2 u \\
&= \kappa_{1,t} \kappa_{2,t} \sum_{k=n}^{\infty} \sum_{m=n}^{\infty} |c_{k+m}(t)|^2 \frac{c[n]_m^2}{c[n]_k^2} (k - n + 1)^2 \\
&= \kappa_{1,t} \kappa_{2,t} \sum_{k=2n}^{\infty} \left( \sum_{j=n}^{k-n} \frac{c[n]_{k-j}^2}{c[n]_j^2} (j - n + 1)^2 \right) |c_k(t)|^2 \\
&= \kappa_{1,t} \kappa_{2,t} \sum_{k=2n}^{\infty} \left( \sum_{j=n}^{k-n} \frac{(k - j - n + 1) \dots (k - j + n - 1)}{(j - n + 1) \dots (j + n - 1)} (j - n + 1)^2 \right) |c_k(t)|^2 \\
&\leq \kappa_{1,t} \kappa_{2,t} \sum_{k=2n}^{\infty} k^{2n} |c_k(t)|^2,
\end{aligned}$$

where

$$\kappa_{1,t} = \|\mathcal{A}_t[n]\|_{\infty}^2, \quad \kappa_{2,t} = \|\mathcal{D}_t[n]\|_{\infty}^2$$

are the squares of the sup-norms of the operators  $\mathcal{A}_t[n]$  and  $\mathcal{D}_t[n]$  respectively. Similarly, we have

(8.5)

$$\begin{aligned} & \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{\beta_n}{2\pi^2 i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) \mathcal{D}_t[n](v, w)}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{v}\zeta)^{2n}(1 - \bar{\eta}\zeta)^2} \frac{g_t''(\eta)}{g_t'(\eta)} \right. \\ & \quad \left. \times \rho(v)^{1-n} \rho(u)^{1-n} d^2 \eta d^2 v d^2 u d\zeta \right|^2 \rho(z)^{1-n} d^2 z \rho(w)^{1-n} d^2 w \\ & \leq \kappa_{1,t} \kappa_{2,t} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi^2 i} \oint_{S^1} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \beta_n}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{v}\zeta)^{2n}(1 - \bar{\eta}\zeta)^2} \frac{g_t''(\eta)}{g_t'(\eta)} d^2 \eta d\zeta \right|^2 \\ & \quad \rho(v)^{1-n} \rho(u)^{1-n} d^2 v d^2 u. \end{aligned}$$

From our result in [TT06], for  $\gamma_t \in S^1 \backslash \text{Diff}_+(S^1)$ ,

$$\varkappa_t = \iint_{\mathbb{D}^*} \left| \frac{g_t''(\eta)}{g_t'(\eta)} \right|^2 d^2 \eta < \infty.$$

It follows from Cauchy-Schwarz inequality that (8.5) is bounded by

$$\begin{aligned} & \leq \kappa_{1,t} \kappa_{2,t} \varkappa_t \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi^2 i} \oint_{S^1} \frac{v_t(\zeta) \beta_n}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{v}\zeta)^{2n}(1 - \bar{\eta}\zeta)^2} d\zeta \right|^2 \\ & \quad d^2 \eta \rho(v)^{1-n} \rho(u)^{1-n} d^2 v d^2 u \\ & = \kappa_{1,t} \kappa_{2,t} \varkappa_t \sum_{k=1}^{\infty} \sum_{m=n}^{\infty} \sum_{l=n}^{\infty} |c_{k+m+l}(t)|^2 \frac{c[1]_k^2 c[n]_m^2}{c[n]_l^2} \\ & \leq \kappa_{1,t} \kappa_{2,t} \varkappa_t \pi^{-1} \sum_{k=2n+1}^{\infty} k^{2n+1} |c_k(t)|^2. \end{aligned}$$

Now we consider the term

$$\begin{aligned} & \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{2n\beta_n}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) \mathcal{D}_t[n](v, w) \bar{v}}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{v}\zeta)^{2n+1}} \rho(u)^{1-n} \rho(v)^{1-n} d^2 u d^2 v d\zeta \right|^2 \\ & \quad \times \rho(z)^{1-n} d^2 z \rho(w)^{1-n} d^2 w. \end{aligned}$$

Using the same reasoning, it is bounded by

$$\begin{aligned} & \kappa_{1,t} \kappa_{2,t} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi i} \oint_{S^1} \frac{2n v_t(\zeta) \bar{v} \beta_n}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{v}\zeta)^{2n+1}} d\zeta \right|^2 \rho(u)^{1-n} d^2 u \rho(v)^{1-n} d^2 v \\ & \leq \kappa_{1,t} \kappa_{2,t} \sum_{m=n}^{\infty} \sum_{l=n}^{\infty} |c_{m+l}(t)|^2 \frac{c[n]_m^2}{c[n]_l^2} (m+n)^2 \\ & \leq \kappa_{1,t} \kappa_{2,t} \sum_{k=2n}^{\infty} k^{2n+2} |c_k(t)|^2. \end{aligned}$$

For the terms containing  $R_t[n](\zeta, w)$ , we have first

$$\begin{aligned}
& \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{1}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) R_t[n](\zeta, w)}{\bar{u}^{2n-2}(\bar{u}\zeta - 1)^2} \rho(u)^{1-n} d^2 u d\zeta \right|^2 \\
& \quad \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w \\
& \leq \kappa_{1,t} \iint_{\mathbb{D}} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) R_t[n](\zeta, w)}{\bar{u}^{2n-2}(\bar{u}\zeta - 1)^2} d\zeta \right|^2 \rho(u)^{1-n} \rho(w)^{1-n} d^2 u d^2 w \\
& \leq (2n-1) \kappa_{1,t} \sum_{l=1-n}^{n-1} \iint_{\mathbb{D}} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) \mathfrak{U}_t[n]_l(w) c[n]_l \zeta^{-l-n}}{\bar{u}^{2n-2}(\bar{u}\zeta - 1)^2} d\zeta \right|^2 \\
& \quad \rho(u)^{1-n} \rho(w)^{1-n} d^2 u d^2 w \\
& = (2n-1) \kappa_{1,t} \alpha_n^{-1} \sum_{l=1-n}^{n-1} c[n]_l^2 \|\mathfrak{U}_t[n]_l\|_{n,2}^2 \left( \sum_{k=n}^{\infty} |c_{k+l}(t)|^2 \frac{(k-n+1)^2}{(k-n+1) \dots (k+n-1)} \right) \\
& \leq (2n-1) \kappa_{1,t} \sum_{l=1-n}^{n-1} \|\mathfrak{U}_t[n]_l\|_{n,2}^2 \sum_{k=1}^{\infty} |c_k(t)|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{1}{2\pi^2 i} \oint_{S^1} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z) R_t[n](\zeta, w) g_t''(\eta)}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{\eta}\zeta)^2} d^2 \eta \rho(u)^{1-n} d^2 u d\zeta \right|^2 \\
& \quad \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w \\
& \leq (2n-1) \kappa_{1,t} \mathfrak{K}_t \sum_{l=1-n}^{n-1} \iint_{\mathbb{D}} \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi^2 i} \oint_{S^1} \frac{v_t(\zeta) \mathfrak{U}_t[n]_l(w) c[n]_l \zeta^{-l-n}}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)(1 - \bar{\eta}\zeta)^2} d\zeta \right|^2 \\
& \quad \times d^2 \eta \rho(u)^{1-n} \rho(w)^{1-n} d^2 u d^2 w \\
& = (2n-1) \frac{\kappa_{1,t} \mathfrak{K}_t}{\alpha_n \pi} \sum_{l=1-n}^{n-1} c[n]_l^2 \|\mathfrak{U}_t[n]_l\|_{n,2}^2 \left( \sum_{k=n}^{\infty} \sum_{m=1}^{\infty} |c_{k+m+l}(t)|^2 \frac{m}{(k-n+1) \dots (k+n-1)} \right) \\
& \leq (2n-1) \frac{\kappa_{1,t} \mathfrak{K}_t}{\pi} \sum_{l=1-n}^{n-1} \|\mathfrak{U}_t[n]_l\|_{n,2}^2 \sum_{k=2}^{\infty} k |c_k(t)|^2.
\end{aligned}$$

Finally,

$$\begin{aligned}
& \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{1}{2\pi i} \oint_{S^1} \iint_{\mathbb{D}^*} \frac{v_t(\zeta) \mathcal{A}_t[n](u, z)}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)} \frac{\partial R_t[n](\zeta, w)}{\partial \zeta} \rho(u)^{1-n} d^2 u d\zeta \right|^2 \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w \\
& \leq (2n-1) \kappa_{1,t} \sum_{l=1-n}^{n-1} \iint_{\mathbb{D}} \iint_{\mathbb{D}^*} \left| \frac{1}{2\pi i} \oint_{S^1} \frac{v_t(\zeta) c[n]_l(l+n) \mathfrak{U}_t[n]_l(w) \zeta^{-l-n-1}}{\bar{u}^{2n-1}(\bar{u}\zeta - 1)} d\zeta \right|^2 \\
& \quad \times \rho(u)^{1-n} \rho(w)^{1-n} d^2 u d^2 w \\
& \leq (2n-1) \kappa_{1,t} \sum_{l=1-n}^{n-1} (l+n)^2 \|\mathfrak{U}_t[n]_l\|_{n,2}^2 \sum_{k=1}^{\infty} |c_k(t)|^2.
\end{aligned}$$

Putting everything together, we find that

$$\iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{d}{ds} \right|_{s=0} \left| \mathcal{C}[n]_{t+s}(z, w) \right|^2 \rho(z)^{1-n} d^2 z \rho(w)^{1-n} d^2 w \leq \mathfrak{K}_t \sum_{k=1}^{\infty} k^{2n+2} |c_k(t)|^2,$$

where

$$\begin{aligned} \mathfrak{K}_t = & \alpha_n^2 \kappa_{1,t} \kappa_{2,t} \left( n^2 + (1-n)^2 + (2n-1)^2 \frac{\mathfrak{K}_t}{\pi} \right) \\ & + (2n-1) \kappa_{1,t} \alpha_n^2 \sum_{l=1-n}^{n-1} \left( n^2 + (1-n)^2 (l+n)^2 + (2n-1)^2 \frac{\mathfrak{K}_t}{\pi} \right) \|\mathfrak{U}_t[n]_l\|_{n,2}^2 \end{aligned}$$

depends continuously on  $t$ , since each of the terms  $\kappa_{1,t}$ ,  $\kappa_{2,t}$ ,  $\mathfrak{K}_t$ ,  $\|\mathfrak{U}_t[n]_l\|_{n,2}^2$  depends smoothly on  $t$ .  $\square$

Now we come back to show that  $\mathcal{C}[n]$  is Hilbert Schmidt. Since  $\mathcal{C}[n] = 0$  when  $\gamma = \text{id}$ , we have

$$\begin{aligned} & \iint_{\mathbb{D}} \iint_{\mathbb{D}} |\mathcal{C}[n](z, w)|^2 \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w \\ &= \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \int_0^1 \frac{d\mathcal{C}[n]_t(z, w)}{dt} dt \right|^2 \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w \\ &\leq \int_0^1 \iint_{\mathbb{D}} \iint_{\mathbb{D}} \left| \frac{d\mathcal{C}[n]_t(z, w)}{dt} \right|^2 \rho(z)^{1-n} \rho(w)^{1-n} d^2 z d^2 w dt \\ &\leq \int_0^1 \mathfrak{K}_t \sum_{k=1}^{\infty} k^{2n+2} |c_k(t)|^2 dt < \infty, \end{aligned}$$

since both  $\mathfrak{K}_t$  and  $\sum_{k=1}^{\infty} k^{2n+2} |c_k(t)|^2$  depend continuously on  $t$ .  $\square$

It follows from Proposition 7.3 and Lemma 7.4 that

**Corollary 8.7.** *For any  $\gamma \in S^1 \backslash \text{Diff}_+(S^1)$  and any  $n \in \mathbb{Z}$ , the matrices  $\Pi_2[\gamma; n]$ ,  $B[\gamma; n]$ ,  $C[\gamma; n]$  define Hilbert-Schmidt operators on  $\ell^2$ .*

Since we have shown in Proposition 8.1 and Proposition 8.2 that  $\mathfrak{S}_1[n]$  and  $C[\gamma; 1-n]^T \overline{C[\gamma; 1-n]}$  are trace class operators, from the identity (7.16)

$$A[\gamma; n] A[\gamma; n]^* = \text{Id} - A[\gamma; n] \mathfrak{S}_1[\gamma^{-1}; n] A[\gamma; n]^* - C[\gamma; 1-n]^T \overline{C[\gamma; 1-n]},$$

we conclude that the Fredholm determinant of  $A[\gamma; n] A[\gamma; n]^*$  is well defined for all  $n \geq 1$ . Now since  $A[\gamma; 1-n] = A[\gamma^{-1}; n]^*$ , the Fredholm determinant of  $A[\gamma; n] A[\gamma; n]^*$  is well defined for all integers  $n$ . On the other hand, since  $\mathfrak{A}[\gamma; n] = A[\gamma; n] \mathfrak{P}[\gamma; n]$ , Corollary 6.5 implies that the Fredholm determinant of  $\mathfrak{A}[\gamma; n] \mathfrak{A}[\gamma; n]^*$  is also well defined for all  $n \geq 1$ . Hence we can make the following definition:

**Definition 8.8.** For any  $n \in \mathbb{Z}$ , we define the real-valued function  $\mathfrak{F}_n : S^1 \backslash \text{Diff}_+(S^1) \rightarrow \mathbb{R}$  by

$$\mathfrak{F}_n = \log \det A[\gamma; n] A[\gamma; n]^*.$$

For any integer  $n \geq 1$ , we define the real-valued function  $\mathfrak{G}_n : S^1 \setminus \text{Diff}_+(S^1) \rightarrow \mathbb{R}$  by

$$\mathfrak{G}_n = \log \det \mathfrak{A}[\gamma; n] \mathfrak{A}[\gamma; n]^* = \log \det (K[n] K[n]^*) = \log \det N_n(\Omega^*).$$

Notice that for  $n \geq 1$ ,

$$\mathfrak{F}_{1-n}(\gamma) = \log \det \mathcal{N}_n(\Omega^*).$$

Some properties of the functions  $\mathfrak{F}_n$  and  $\mathfrak{G}_n$  are listed below.

**Proposition 8.9.** *We have the followings:*

- A.** For any integer  $n$ ,  $\mathfrak{F}_n(\gamma) = \mathfrak{F}_{1-n}(\gamma^{-1})$ .
- B.** For any integer  $n \geq 1$ ,  $\mathfrak{G}_n$  is invariant with respect to the inversion  $\mathfrak{I}$  on  $S^1 \setminus \text{Diff}_+(S^1)$ , i.e.,  $\mathfrak{G}_n(\gamma) = \mathfrak{G}_n(\gamma^{-1})$ .
- C.** For any integer  $n \geq 1$ ,  $\mathfrak{F}_n = \mathfrak{G}_n = \mathfrak{F}_{1-n}$ .
- D.** For any integer  $n$ ,

$$\mathfrak{F}_n(\gamma) = -\log \det \Pi_1[\gamma; n] \Pi_1[\gamma; n]^* = -\log \det \Pi_1[\gamma; 1-n] \Pi_1[\gamma; 1-n]^*.$$

- E.** Both the functions  $\mathfrak{F}_n$  and  $\mathfrak{G}_n$  are constant on each fiber of the manifold  $S^1 \setminus \text{Diff}_+(S^1)$  over  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ . Hence they descend to well-defined functions on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , which we denote by the same symbols.

*Proof.* Notice that for any operator  $K$  where  $KK^* - \text{id}$  is a trace class operator,  $\det KK^* = \det K^*K$ . **A.** follows from the identity

$$A[\gamma; 1-n] = A[\gamma^{-1}; n]^*$$

and **B.** follows from

$$\mathfrak{A}[\gamma^{-1}; n] = \overline{\mathfrak{D}[\gamma; n]} = \mathfrak{A}[\gamma; n]^*.$$

It is well known in Fredholm theory that if both  $K_1$  and  $K_2$  are operators such that  $K_1 - \text{id}$  and  $K_2 - \text{id}$  are trace class operators, then  $\det K_1 K_2 = \det K_1 \det K_2 = \det K_2 K_1$ . On the other hand, since  $\mathfrak{P}[\gamma; n]$  is an upper triangular matrix with diagonal elements all equal to 1,  $\det \mathfrak{P}[\gamma; n] = \det \mathfrak{P}[\gamma; n]^* = 1$ . It follows that

$$\begin{aligned} \det \mathfrak{A}[\gamma; n] \mathfrak{A}[\gamma; n]^* &= \det \mathfrak{A}[\gamma; n]^* \mathfrak{A}[\gamma; n] \\ &= \det (\mathfrak{P}[\gamma; n]^* A[\gamma; n]^* A[\gamma; n] \mathfrak{P}[\gamma; n]) \\ &= \det (\mathfrak{P}[\gamma; n]^*) \det (A[\gamma; n]^* A[\gamma; n]) \det (\mathfrak{P}[\gamma; n]) \\ &= \det (A[\gamma; n] A[\gamma; n]^*), \end{aligned}$$

i.e.  $\mathfrak{G}_n = \mathfrak{F}_n$ . We can then conclude the other equality in **C.** by **A.** and **B.**

Now, since  $A[\gamma^{-1}; n] = \Pi_1[\gamma; n]^{-1}$ , by **B.** and **C.**,

$$\mathfrak{F}_n(\gamma) = \mathfrak{F}_n(\gamma^{-1}) = \log \det (A[\gamma^{-1}; n] A[\gamma^{-1}; n]^*) = -\log \det (\Pi_1[\gamma; n] \Pi_1[\gamma; n]^*).$$

Together with **C.** prove **D.**

Finally, when  $\sigma \in \text{PSU}(1, 1)$  is a linear fractional transformation,  $\mathcal{S}(\sigma) = 0$ . Therefore,  $\mathfrak{G}[\sigma; n] = 0$ . On the other hand, since  $\sigma$  extends to a holomorphic function on  $\mathbb{D}$ , we have

$$(8.6) \quad \hat{\Pi}_2[\sigma; n] = 0 \quad \text{for all } n \geq 1.$$

It follows from (7.12) that

$$(8.7) \quad \Pi_1[\sigma; n] \Pi_1[\sigma; n]^* = \text{Id}.$$

By (7.1)

$$\Pi[\sigma \circ \gamma_0] = \Pi[\gamma_0; n] \Pi[\sigma; n]$$

and (8.6), we conclude that

$$(8.8) \quad \Pi_1[\sigma \circ \gamma_0] = \Pi_1[\gamma_0; n] \Pi_1[\sigma; n].$$

It follows from **D.**, (8.7) and (8.8) that

$$\begin{aligned} \mathfrak{F}_n(\sigma \circ \gamma_0) &= -\log \det (\Pi_1[\sigma \circ \gamma_0; n] \Pi_1[\sigma \circ \gamma_0; n]^*) \\ &= -\log \det (\Pi_1[\gamma_0; n] \Pi_1[\gamma_0; n]^*) = \mathfrak{F}_n(\gamma_0). \end{aligned}$$

Therefore,  $\mathfrak{F}_n$  is invariant on each fiber of  $S^1 \setminus \text{Diff}_+(S^1) \rightarrow \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ . By **C.**, the same holds for  $\mathfrak{G}_n$ .  $\square$

## 9. FIRST AND SECOND DERIVATIVES OF THE FUNCTIONS $\mathfrak{F}_n$ AND $\mathfrak{G}_n$ ON $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$

In this section, we compute the first derivatives of the function  $\mathfrak{F}_n = \mathfrak{G}_n$ . We begin by an interesting lemma.

**Lemma 9.1.** *Let  $E$  be a domain and  $h : E \rightarrow \hat{\mathbb{C}}$  a univalent function. Then we have the following formula:*

$$\lim_{w \rightarrow z} \left( \left( n \frac{\partial}{\partial z} + (n-1) \frac{\partial}{\partial w} \right) \left[ \frac{h'(z)^{1-n} h'(w)^n}{h(z) - h(w)} - \frac{1}{z - w} \right] \right) = -\frac{6n^2 - 6n + 1}{6} \mathcal{S}(h)(z).$$

The proof is just some calculus. However, this formula is essential in our theorem below. I am grateful to A. McIntyre who pointed out this formula to me a few years ago.

**Theorem 9.2.** *Let  $\gamma \in \text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  and  $\dot{\gamma} = (v, \bar{v})$  a tangent vector at  $\gamma$ . The first derivative of the function  $\mathfrak{F}_n$  is given by*

$$\begin{aligned} \partial \mathfrak{F}_n(v) &= \frac{6n^2 - 6n + 1}{12\pi i} \oint_{S^1} \mathcal{S}(g)(z) v(z) dz \\ \bar{\partial} \mathfrak{F}_n(\bar{v}) &= -\frac{6n^2 - 6n + 1}{12\pi i} \oint_{S^1} \overline{\mathcal{S}(g)(z)} \bar{v}(z) d\bar{z}. \end{aligned}$$

*Proof.* We use the definition

$$\mathfrak{F}_n(\gamma) = -\log \Pi_1[\gamma; 1-n] \Pi_1[\gamma; 1-n]^*$$

for the function  $\mathfrak{F}_n$ .

Let  $\gamma_t$  be a smooth curve in  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  which defines the tangent vector  $\dot{\mathbf{u}} = (\mathbf{v}, \bar{\mathbf{v}})$  at  $\gamma_0 = \gamma$ . It is a standard fact that

$$\begin{aligned}
 (9.1) \quad & \left. \frac{d}{dt} \right|_{t=0} \log \det(\Pi_1[\gamma_t; 1-n] \Pi_1[\gamma_t; 1-n]^*) \\
 &= \text{Tr} \left( (\Pi_1[\gamma; 1-n] \Pi_1[\gamma; 1-n]^*)^{-1} \left. \frac{d}{dt} \right|_{t=0} (\Pi_1[\gamma_t; 1-n] \Pi_1[\gamma_t; 1-n]^*) \right) \\
 &= \text{Tr} \left( (\Pi_1[\gamma; 1-n]^*)^{-1} \Pi_1[\gamma; 1-n]^{-1} \left( \left. \frac{d}{dt} \right|_{t=0} \Pi_1[\gamma_t; 1-n] \right) \Pi_1[\gamma; 1-n]^* \right. \\
 &\quad \left. + (\Pi_1[\gamma; 1-n]^*)^{-1} \left( \left. \frac{d}{dt} \right|_{t=0} \Pi_1[\gamma_t; 1-n]^* \right) \right).
 \end{aligned}$$

Let  $\mathbf{u}_t = \gamma_t \circ \gamma^{-1}$  and

$$\dot{\mathbf{u}}(z) = \left. \frac{d\mathbf{u}_t}{dt} \right|_{t=0} (z) = \sum_{k \in \mathbb{Z}} c_k z^{k+1}.$$

By (7.1), we have

$$\Pi[\gamma_t; 1-n] = \Pi[\gamma; 1-n] \Pi[\mathbf{u}_t; 1-n].$$

Therefore,

$$(9.2) \quad \left. \frac{d}{dt} \right|_{t=0} \Pi[\gamma_t; 1-n] = \Pi[\gamma; 1-n] \left. \frac{d}{dt} \right|_{t=0} \Pi[\mathbf{u}_t; 1-n].$$

By definition,

$$c[1-n]_k \mathbf{u}_t(z)^{k+n-1} \mathbf{u}'_t(z)^{1-n} = \sum_{l \in \mathbb{Z}} \Pi[\mathbf{u}_t; 1-n]_{lk} c[1-n]_l z^{l+n-1}, \quad z \in S^1.$$

Differentiate both sides with respect to  $t$ , we have

$$\begin{aligned}
 & \sum_{l \in \mathbb{Z}} \left( \left. \frac{d}{dt} \right|_{t=0} \Pi[\mathbf{u}_t; 1-n]_{lk} \right) c[1-n]_l z^{l+n-1} \\
 &= c[1-n]_k \left( (k+n-1) z^{k+n-2} \dot{\mathbf{u}}(z) + (1-n) z^{k+n-1} \dot{\mathbf{u}}'(z) \right) \\
 &= c[1-n]_k \sum_{l \in \mathbb{Z}} (k+n-1 + (1-n)(l+1)) c_l z^{l+k+n-1}.
 \end{aligned}$$

Comparing both sides, we find that

$$\begin{aligned}
 \left. \frac{d}{dt} \right|_{t=0} \Pi[\mathbf{u}_t; 1-n]_{lk} &= \frac{c[1-n]_k}{c[1-n]_l} (k+n-1 + (1-n)(l-k+1)) c_{l-k} \\
 &= \frac{1}{\alpha_n} c[1-n]_k c[n]_l (nk + (1-n)l) c_{l-k}.
 \end{aligned}$$

From (9.2) again, we have

$$\left. \frac{d}{dt} \right|_{t=0} \Pi_1[\gamma_t; 1-n] = \Pi_1[\gamma; 1-n] \left. \frac{d}{dt} \right|_{t=0} \Pi_1[\mathbf{u}_t; 1-n] + \hat{\Pi}_3[\gamma; 1-n] \left. \frac{d}{dt} \right|_{t=0} \hat{\Pi}_2[\mathbf{u}_t; 1-n].$$



Therefore, (9.1) is equal to

$$\begin{aligned}
& \text{Tr} \left( (\Pi_1[\gamma; 1-n]^*)^{-1} \left( \frac{d}{dt} \Big|_{t=0} \Pi_1[u_t; 1-n] + \frac{d}{dt} \Big|_{t=0} \Pi_1[u_t; 1-n]^* \right) \Pi_1[\gamma; 1-n]^* \right. \\
& + (\Pi_1[\gamma; 1-n]^*)^{-1} \Pi_1[\gamma; 1-n]^{-1} \hat{\Pi}_3[\gamma; 1-n] \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n] \right) \Pi_1[\gamma; 1-n]^* \\
& \left. + (\Pi_1[\gamma; 1-n]^*)^{-1} \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n]^* \right) \hat{\Pi}_3[\gamma; 1-n]^* \right) \\
& = \text{Tr} \left( \frac{d}{dt} \Big|_{t=0} \Pi_1[u_t; 1-n] + \frac{d}{dt} \Big|_{t=0} \Pi_1[u_t; 1-n]^* \right. \\
& + (\Pi_1[\gamma; 1-n]^{-1} \hat{\Pi}_3[\gamma; 1-n] \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n] \right) \\
& \left. + \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n]^* \right) \hat{\Pi}_3[\gamma; 1-n]^* (\Pi_1[\gamma; 1-n]^*)^{-1} \right).
\end{aligned}$$

Now, since  $c_0$  is purely imaginary, and for any  $k$ ,

$$\frac{d}{dt} \Big|_{t=0} \Pi_1[u_t; 1-n]_{kk} = kc_0,$$

we have

$$\text{Tr} \left( \frac{d}{dt} \Big|_{t=0} \Pi_1[\gamma_t; 1-n] + \frac{d}{dt} \Big|_{t=0} \Pi_1[\gamma_t; 1-n]^* \right) = 0.$$

On the other hand, by Lemma 7.2 and Proposition 7.3,

$$\hat{\Pi}_3[\gamma; 1-n]^* (\Pi_1[\gamma; 1-n]^*)^{-1} = \hat{\Pi}_2[\gamma^{-1}; n] \Pi_1[\gamma^{-1}; n]^{-1} = \hat{B}[\gamma; n].$$

Therefore,

$$\begin{aligned}
& \text{Tr} \left( \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n]^* \right) \hat{\Pi}_3[\gamma; 1-n]^* (\Pi_1[\gamma; 1-n]^*)^{-1} \right) \\
& = \sum_{l=1-n}^{\infty} \sum_{k=n}^{\infty} B[\gamma; n]_{lk} \left( \frac{d}{dt} \Big|_{t=0} \overline{\hat{\Pi}_2[u_t; 1-n]_{lk}} \right) \\
& = \frac{1}{\alpha_n} \sum_{l=1-n}^{\infty} \sum_{k=n}^{\infty} B[\gamma; n]_{lk} c[1-n]_k c[n]_l (nk + (n-1)l) \overline{c_{-l-k}} \\
& = -\frac{1}{\alpha_n} \sum_{l=1-n}^{\infty} \sum_{k=n}^{\infty} B[\gamma; n]_{lk} c[1-n]_k c[n]_l (nk + (n-1)l) c_{l+k} \\
& = -\frac{1}{2\pi i} \oint_{S^1} T(z) \tilde{v}(z) dz,
\end{aligned}$$

where

$$\tilde{v}(z) = \sum_{k=1}^{\infty} c_k z^{k+1},$$

and

$$\begin{aligned}
T(z) &= \frac{1}{\alpha_n} \sum_{l=1-n}^{\infty} \sum_{k=n}^{\infty} B[\gamma; n]_{lk} c[1-n]_k c[n]_l (nk + (n-1)l) z^{-l-k-2} \\
&= - \lim_{w \rightarrow z} \left( \left( n \frac{\partial}{\partial z} + (n-1) \frac{\partial}{\partial w} \right) \left[ \frac{1}{\alpha_n} \sum_{l=1-n}^{\infty} \sum_{k=n}^{\infty} B[\gamma; n]_{lk} c[1-n]_k c[n]_l w^{-l-n} z^{-k+n-1} \right] \right) \\
&= - \lim_{w \rightarrow z} \left( \left( n \frac{\partial}{\partial z} + (n-1) \frac{\partial}{\partial w} \right) \left[ \frac{g'(z)^{1-n} g'(w)^n}{g(z) - g(w)} - \frac{1}{z-w} \right] \right) \\
&= \frac{6n^2 - 6n + 1}{6} \mathcal{S}(g)(z).
\end{aligned}$$

Since  $\mathcal{S}(g)(z) = O(z^{-4})$  as  $z \rightarrow \infty$ , the  $c_1$  term in  $\tilde{v}(z)$  does not contribute to the integral

$$\oint_{S^1} T(z) \tilde{v}(z) dz.$$

Therefore, we can replace  $\tilde{v}$  by  $v = \sum_{k=2}^{\infty} c_k z^{k+1}$  and we have shown that

$$\begin{aligned}
&\text{Tr} \left( \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n]^* \right) \hat{\Pi}_3[\gamma; 1-n]^* (\Pi_1[\gamma; 1-n]^*)^{-1} \right) \\
&= - \frac{6n^2 - 6n + 1}{12\pi i} \oint_{S^1} \mathcal{S}(g)(z) v(z) dz.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\text{Tr} \left( \Pi_1[\gamma; 1-n]^{-1} \hat{\Pi}_3[\gamma; 1-n] \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n] \right) \right) \\
&= \text{Tr} \left( \left( \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n]^* \right) \hat{\Pi}_3[\gamma; 1-n]^* (\Pi_1[\gamma; 1-n]^*)^{-1} \right)^* \right) \\
&= \overline{\text{Tr} \left( \left( \frac{d}{dt} \Big|_{t=0} \hat{\Pi}_2[u_t; 1-n]^* \right) \hat{\Pi}_3[\gamma; 1-n]^* (\Pi_1[\gamma; 1-n]^*)^{-1} \right)} \\
&= \frac{6n^2 - 6n + 1}{12\pi i} \oint_{S^1} \overline{\mathcal{S}(g)(z)} \bar{v}(z) d\bar{z}.
\end{aligned}$$

Combining together prove the assertion of the theorem.  $\square$

In [SH62] (see also [TT06]), the function  $S : \text{Möb}(S^1) \setminus \text{Diff}_+(S^1) \rightarrow \mathbb{R}$  defined by

$$S(\gamma) = \iint_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 d^2 z + \iint_{\mathbb{D}^*} \left| \frac{g''(z)}{g'(z)} \right|^2 d^2 z - 4\pi \log |g'(\infty)|$$

was shown to satisfy

$$\partial S(v) = i \oint_{S^1} \mathcal{S}(g)(z) v(z) dz$$

and

$$\partial \bar{\partial} S(v, \bar{v}) = \|v\|^2.$$

The later implies that  $S$  is a Weil-Petersson potential on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ . Since  $\Pi_1[\text{id}; n] = \text{id}$  for all  $n \in \mathbb{Z}$ ,  $\mathfrak{F}_n(\text{id}) = 0$ . Together with the fact that  $S(\text{id}) = 0$ , we conclude that

**Corollary 9.3.** *On  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , we have the following:*

**I.**

$$\mathfrak{F}_n = -\frac{6n^2 - 6n + 1}{12\pi} S.$$

**II.** *Universal Index Theorem on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  I.*

$$\partial\bar{\partial}\mathfrak{F}_n(v, \bar{v}) = -\frac{6n^2 - 6n + 1}{12\pi} \|v\|^2.$$

This proves our main result.

**Theorem 9.4.** *Universal Index Theorem on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$  II.*

**I.** *For every point on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , the determinant of period matrix of holomorphic  $n$ -differentials  $N_n$  is related to the Weil-Petersson potential  $S$  by*

$$\det N_n = \exp\left(-\frac{6n^2 - 6n + 1}{12\pi} S\right).$$

**II.** *For every point on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , the determinant of the period matrix of holomorphic  $n$ -differentials  $N_n$  is related to the period matrix of holomorphic one forms  $N_1$  by*

$$\det N_n = (\det N_1)^{6n^2 - 6n + 1}.$$

The item **II** of the theorem is the universal version of Mumford's isomorphism [Mum77]  $\lambda_n = \lambda_1^{\otimes 6n^2 - 6n + 1}$ , where  $\lambda_n$  is the determinant line bundle of  $n$ -tensor power of the vertical cotangent bundle of the fibration  $\mathcal{C}_g \rightarrow \mathcal{M}_g$ , i.e. the fibration of the universal curve over the moduli space of Riemann surfaces of genus  $g$ .

On the other hand, we can also interpret our result in terms of the Bers integral operator.

**Theorem 9.5.** *For every point on  $\text{Möb}(S^1) \setminus \text{Diff}_+(S^1)$ , the Bers integral operator  $K[n]$ ,  $n \geq 1$  satisfies*

$$\det(K[n]K[n]^*) = \exp\left(-\frac{6n^2 - 6n + 1}{12\pi} S\right) = [\det(K[1]K[1]^*)]^{6n^2 - 6n + 1}.$$

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## APPENDIX A. A CONJECTURE

As we mentioned in the introduction, the Bers integral operator  $K[1]$  is a bounded operator with norm less than or equal to one. We conjecture that for all  $n \geq 2$ ,  $K[n]$  is also a bounded operator of norm less than or equal to one. Below we give a partial support to this conjecture.

**Lemma A.1.** *If  $\phi \in A_{1,2}(\mathbb{D})$ , then  $\phi^n \in A_{n,2}(\mathbb{D})$ .*

*Proof.* Since  $\phi \in A_{1,2}(\mathbb{D})$ , by Lemma 1.3 in Chapter 2 of [TT06],

$$\|\phi\|_{1,\infty} = \sup_{z \in \mathbb{D}} \rho(z)^{-1/2} |\phi(z)| < \infty.$$

Therefore

$$\iint_{\mathbb{D}} \rho(z)^{1-n} |\phi^n(z)|^2 d^2 z \leq \|\phi\|_{\infty}^{2n-2} \iint_{\mathbb{D}} |\phi(z)|^2 d^2 z < \infty.$$

□

**Lemma A.2.** *Let  $n \geq 2$  be an integer. If  $\phi \in A_{1,2}(\mathbb{D})$ , then*

$$\|\phi^n\|_{n,2}^2 \leq \frac{\beta_1 \beta_{n-1}}{\beta_n} \|\phi\|_{1,2}^2 \|\phi^{n-1}\|_{n-1,2}^2.$$

*Proof.* Let

$$\begin{aligned} \phi(z) &= \sum_{k=1}^{\infty} a_k c[1]_k z^{k-1}, \\ \phi^{n-1}(z) &= \sum_{k=n-1}^{\infty} b_k c[n-1]_k z^{k-n+1}, \\ \phi^n(z) &= \sum_{k=n}^{\infty} c_k c[n]_k z^{k-n}. \end{aligned}$$

Then

$$c[n]_k c_k = \sum_{l=n-1}^{k-1} c[1]_{k-l} c[n-1]_l a_{k-l} b_l.$$

We have

$$\begin{aligned} \|\phi^n\|_{n,2}^2 &= \sum_{k=n}^{\infty} |c_k|^2 = \sum_{k=n}^{\infty} \frac{1}{c[n]_k^2} \left| \sum_{l=n-1}^{k-1} c[1]_{k-l} c[n-1]_l a_{k-l} b_l \right|^2 \\ &\leq \sum_{k=n}^{\infty} \frac{1}{c[n]_k^2} \left( \sum_{l=n-1}^{k-1} c[1]_{k-l}^2 c[n-1]_l^2 \right) \left( \sum_{l=n-1}^{k-1} |a_{k-l}|^2 |b_l|^2 \right) \end{aligned}$$

From the identities

$$\frac{\beta_m}{(1-x)^{2m}} = \sum_{k=m}^{\infty} c[m]_k^2 x^{k-m}$$

and

$$\frac{1}{(1-x)^2} \frac{1}{(1-x)^{2n-2}} = \frac{1}{(1-x)^{2n}},$$

we find that

$$\frac{1}{\beta_1 \beta_{n-1}} \sum_{l=n-1}^{k-1} c[1]_{k-l}^2 c[1-n]_l^2 = \frac{1}{\beta_n} c[n]_k^2.$$

Therefore

$$\begin{aligned} \|\phi^n\|_{n,2}^2 &\leq \frac{\beta_1 \beta_{n-1}}{\beta_n} \sum_{k=n}^{\infty} \sum_{l=n-1}^{k-1} |a_{k-l}|^2 |b_l|^2 \\ &= \frac{\beta_1 \beta_{n-1}}{\beta_n} \left( \sum_{k=1}^{\infty} |a_k|^2 \right) \left( \sum_{l=n-1}^{\infty} |b_l|^2 \right) \\ &= \frac{\beta_1 \beta_{n-1}}{\beta_n} \|\phi\|_{1,2}^2 \|\phi\|_{n-1,2}^2. \end{aligned}$$

□

The following is the main support of our conjecture.

**Theorem A.3.** *Let  $n$  be an integer. Then*

$$(K[n]K[n]^*)(z, z) = \iint_{\mathbb{D}^*} |K[n](z, w)|^2 \rho(w)^{1-n} d^2 w \leq \text{Id}[n](z, z).$$

*Proof.* Let  $\phi_z \in A_{1,2}(\mathbb{D}^*)$  be defined as

$$\phi_z(w) = K[1](z, w) = \beta_1 \frac{f'(z)g'(w)}{(f(z) - g(w))^2}.$$

Then

$$K[n](z, w) = \beta_n \left( \frac{f'(z)g'(w)}{(f(z) - g(w))^2} \right)^n = \frac{\beta_n}{\beta_1^n} \phi_z^n(w).$$

We prove the theorem by induction. The case  $n = 1$  is known to be true (see [TT06]). For  $n \geq 2$ , suppose that

$$(K[n-1]K[n-1]^*)(z, z) \leq \text{Id}[n-1](z, z) = \frac{\beta_{n-1}}{(1 - |z|^2)^{2n-2}},$$

then

$$\|\phi_z^{n-1}\|_{n-1,2}^2 = \frac{\beta_1^{2n-2}}{\beta_{n-1}^2} (K[n-1]K[n-1]^*)(z, z) \leq \frac{\beta_1^{2n-2}}{\beta_{n-1}} \frac{1}{(1 - |z|^2)^{2n-2}}.$$

Therefore by Lemma A.2,

$$\begin{aligned} (K[n]K[n]^*)(z, z) &= \frac{\beta_n^2}{\beta_1^{2n}} \|\phi_z^n\|_{n,2}^2 \leq \frac{\beta_n^2}{\beta_1^{2n}} \frac{\beta_1 \beta_{n-1}}{\beta_n} \|\phi_z\|_{1,2}^2 \|\phi_z^{n-1}\|_{n-1,2}^2 \\ &\leq \frac{\beta_n \beta_{n-1}}{\beta_1^{2n-1}} \frac{\beta_1^{2n-2} \beta_1}{\beta_{n-1}} \frac{1}{(1 - |z|^2)^{2n}} \\ &= \frac{\beta_n}{(1 - |z|^2)^{2n}} = \text{Id}[n](z, z). \end{aligned}$$

By induction, the theorem is true for all  $n \geq 1$ .

□

This theorem implies that the kernel of the operator  $(\text{Id}[n] - K[n]K[n]^*)$  satisfies  $(\text{Id}[n] - K[n]K[n]^*)(z, z) \geq 0$ . Therefore, we conjecture that  $(\text{Id}[n] - K[n]K[n]^*)$  is a positive definite operator, which means that  $K[n]$  is an operator with norm less than or equal to one.

## REFERENCES

- [Ber66] Lipman Bers, *A non-standard integral equation with applications to quasiconformal mappings*, Acta Math. **116** (1966), 113–134.
- [Dur83] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 259, Springer-Verlag, New York, 1983.
- [Hum72] J. A. Hummel, *Inequalities of Grunsky type for Aharonov pairs*, J. Analyse Math. **25** (1972), 217–257.
- [Kir87] A. A. Kirillov, *Kähler structure on the  $K$ -orbits of a group of diffeomorphisms of the circle*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 2, 42–45.
- [KY88] A. A. Kirillov and D. V. Yuriev, *Representations of the Virasoro algebra by the orbit method*, J. Geom. Phys. **5** (1988), no. 3, 351–363.
- [MT06a] Andrew McIntyre and Leon A. Takhtajan, *Holomorphic factorization of determinants of Laplacians on Riemann surfaces and a higher genus generalization of Kronecker’s first limit formula*, preprint arXiv math.CV/0410294, to appear in Geom. and Funct. Analysis **16** (2006).
- [MT06b] Andrew McIntyre and Lee-Peng Teo, *Holomorphic factorization of determinants of laplacians using quasi-fuchsian uniformization*, preprint arXiv math.CV/0605605 (2006).
- [Mum77] David Mumford, *Stability of projective varieties*, Enseignement Math. (2) **23** (1977), no. 1-2, 39–110.
- [Nag88] Subhashis Nag, *The complex analytic theory of Teichmüller spaces*, John Wiley & Sons Inc., New York, 1988, A Wiley-Interscience Publication.
- [Nag92] ———, *A period mapping in universal Teichmüller space*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), no. 2, 280–287.
- [NS95] Subhashis Nag and Dennis Sullivan, *Teichmüller theory and the universal period mapping via quantum calculus and the  $H^{1/2}$  space on the circle*, Osaka J. Math. **32** (1995), no. 1, 1–34.
- [Pom75] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975, With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.
- [Sch57] M. Schiffer, *The Fredholm eigen values of plane domains*, Pacific J. Math. **7** (1957), 1187–1225.
- [SH62] M. Schiffer and N. S. Hawley, *Connections and conformal mapping*, Acta Math. **107** (1962), 175–274.
- [Teo03] Lee-Peng Teo, *Analytic functions and integrable hierarchies—characterization of tau functions*, Lett. Math. Phys. **64** (2003), no. 1, 75–92.
- [Teo04] ———, *The Velling-Kirillov metric on the universal Teichmüller curve*, J. Anal. Math. **93** (2004), 271–307.
- [TT06] L.-P. Teo and L.A. Takhtajan, *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc. **183** (2006), no. 861, vi+119.
- [TZ91] L. A. Takhtajan and P. G. Zograf, *A local index theorem for families of  $\bar{\partial}$ -operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces*, Comm. Math. Phys. **137** (1991), no. 2, 399–426.

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